

CHAPTER 13 INTEGRATION IN VECTOR FIELDS

13.1 LINE INTEGRALS

1. $\mathbf{r} = t\mathbf{i} + (1-t)\mathbf{j} \Rightarrow x = t$ and $y = 1 - t \Rightarrow y = 1 - x \Rightarrow (c)$

2. $\mathbf{r} = \mathbf{i} + \mathbf{j} + t\mathbf{k} \Rightarrow x = 1$, $y = 1$, and $z = t \Rightarrow (e)$

3. $\mathbf{r} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} \Rightarrow x = 2 \cos t$ and $y = 2 \sin t \Rightarrow x^2 + y^2 = 4 \Rightarrow (g)$

4. $\mathbf{r} = t\mathbf{i} \Rightarrow x = t$, $y = 0$, and $z = 0 \Rightarrow (a)$

5. $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \Rightarrow x = t$, $y = t$, and $z = t \Rightarrow (d)$

6. $\mathbf{r} = t\mathbf{j} + (2 - 2t)\mathbf{k} \Rightarrow y = t$ and $z = 2 - 2t \Rightarrow z = 2 - 2y \Rightarrow (b)$

7. $\mathbf{r} = (t^2 - 1)\mathbf{j} + 2t\mathbf{k} \Rightarrow y = t^2 - 1$ and $z = 2t \Rightarrow y = \frac{z^2}{4} - 1 \Rightarrow (f)$

8. $\mathbf{r} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{k} \Rightarrow x = 2 \cos t$ and $z = 2 \sin t \Rightarrow x^2 + z^2 = 4 \Rightarrow (h)$

9. $\mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{j}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{2}\mathbf{j}$; $x = t$ and $y = 1 - t \Rightarrow x + y = t + (1 - t) = 1$

$$\Rightarrow \int_C f(x, y, z) ds = \int_0^1 f(t, 1-t, 0) \frac{d\mathbf{r}}{dt} dt = \int_0^1 (1)(\sqrt{2}) dt = [\sqrt{2}t]_0^1 = \sqrt{2}$$

10. $\mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{j} + \mathbf{k}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{2}$; $x = t$, $y = 1 - t$, and $z = 1 \Rightarrow x - y + z = 2$

$$= t - (1 - t) + 1 - 2 = 2t - 2 \Rightarrow \int_C f(x, y, z) ds = \int_0^1 (2t - 2)\sqrt{2} dt = \sqrt{2}[t^2 - 2t]_0^1 = -\sqrt{2}$$

11. $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + (2 - 2t)\mathbf{k}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{4 + 1 + 4} = 3$; $xy + y + z$

$$= (2t)t + t + (2 - 2t) \Rightarrow \int_C f(x, y, z) ds = \int_0^1 (2t^2 - t + 2) 3 dt = 3 \left[\frac{2}{3}t^3 - \frac{1}{2}t^2 + 2t \right]_0^1 = 3 \left(\frac{2}{3} - \frac{1}{2} + 2 \right) = \frac{13}{2}$$

12. $\mathbf{r}(t) = (4 \cos t)\mathbf{i} + (4 \sin t)\mathbf{j} + 3t\mathbf{k}$, $-2\pi \leq t \leq 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-4 \sin t)\mathbf{i} + (4 \cos t)\mathbf{j} + 3\mathbf{k}$

$$\Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{16 \sin^2 t + 16 \cos^2 t + 9} = 5; \sqrt{x^2 + y^2} = \sqrt{16 \cos^2 t + 16 \sin^2 t} = 4 \Rightarrow \int_C f(x, y, z) ds = \int_{-2\pi}^{2\pi} (4)(5) dt$$

$$= [20t]_{-2\pi}^{2\pi} = 80\pi$$

$$13. \mathbf{r}(t) = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + t(-\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = (1-t)\mathbf{i} + (2-3t)\mathbf{j} + (3-2t)\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$$

$$\Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1+9+4} = \sqrt{14}; x+y+z = (1-t) + (2-3t) + (3-2t) = 6-6t \Rightarrow \int_C f(x,y,z) ds$$

$$= \int_0^1 (6-6t) \sqrt{14} dt = 6\sqrt{14} \left[1 - \frac{t^2}{2} \right]_0^1 = (6\sqrt{14}) \left(\frac{1}{2} \right) = 3\sqrt{14}$$

$$14. \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, 1 \leq t \leq \infty \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{3}; \frac{\sqrt{3}}{x^2 + y^2 + z^2} = \frac{\sqrt{3}}{t^2 + t^2 + t^2} = \frac{\sqrt{3}}{3t^2}$$

$$\Rightarrow \int_C f(x,y,z) ds = \int_1^\infty \left(\frac{\sqrt{3}}{3t^2} \right) \sqrt{3} dt = \left[-\frac{1}{t} \right]_1^\infty = \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + 1 \right) = 1$$

$$15. C_1: \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1+4t^2}; x + \sqrt{y-z^2} = t + \sqrt{t^2} - 0 = t + |t| = 2t$$

$$\Rightarrow \int_{C_1} f(x,y,z) ds = \int_0^1 2t \sqrt{1+4t^2} dt = \left[\frac{1}{6}(1+4t^2)^{3/2} \right]_0^1 = \frac{1}{6}(5)^{3/2} - \frac{1}{6} = \frac{1}{6}(5\sqrt{5} - 1);$$

$$C_2: \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1; x + \sqrt{y-z^2} = 1 + \sqrt{1-t^2} = 2 - t^2$$

$$\Rightarrow \int_{C_2} f(x,y,z) ds = \int_0^1 (2-t^2)(1) dt = \left[2t - \frac{1}{3}t^3 \right]_0^1 = 2 - \frac{1}{3} = \frac{5}{3}; \text{ therefore } \int_C f(x,y,z) ds$$

$$= \int_{C_1} f(x,y,z) ds + \int_{C_2} f(x,y,z) ds = \frac{5}{6}\sqrt{5} + \frac{3}{2}$$

$$16. C_1: \mathbf{r}(t) = t\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1; x + \sqrt{y-z^2} = 0 + \sqrt{0-t^2} = -t^2$$

$$\Rightarrow \int_{C_1} f(x,y,z) ds = \int_0^1 (-t^2)(1) dt = \left[-\frac{t^3}{3} \right]_0^1 = -\frac{1}{3};$$

$$C_2: \mathbf{r}(t) = t\mathbf{j} + \mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1; x + \sqrt{y-z^2} = 0 + \sqrt{t-1} = \sqrt{t-1}$$

$$\Rightarrow \int_{C_2} f(x,y,z) ds = \int_0^1 (\sqrt{t-1})(1) dt = \left[\frac{2}{3}t^{3/2} - t \right]_0^1 = \frac{2}{3} - 1 = -\frac{1}{3};$$

$$C_3: \mathbf{r}(t) = t\mathbf{i} + \mathbf{j} + \mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 1; x + \sqrt{y-z^2} = t + \sqrt{1-t^2} = t$$

$$\Rightarrow \int_{C_3} f(x,y,z) ds = \int_0^1 (t)(1) dt = \left[\frac{t^2}{2} \right]_0^1 = \frac{1}{2} \Rightarrow \int_C f(x,y,z) ds = \int_{C_1} f ds + \int_{C_2} f ds + \int_{C_3} f ds = -\frac{1}{3} + \left(-\frac{1}{3} \right) + \frac{1}{2}$$

$$= -\frac{1}{6}$$

$$17. \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, 0 < a \leq t \leq b \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{3}; \frac{x+y+z}{x^2+y^2+z^2} = \frac{t+t+t}{t^2+t^2+t^2} = \frac{1}{t}$$

$$\Rightarrow \int_C f(x, y, z) ds = \int_a^b \left(\frac{1}{t} \right) \sqrt{3} dt = [\sqrt{3} \ln |t|]_a^b = \sqrt{3} \ln \left(\frac{b}{a} \right), \text{ since } 0 < a \leq b$$

$$18. \mathbf{r}(t) = (a \cos t)\mathbf{j} + (a \sin t)\mathbf{k}, 0 \leq t \leq 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-a \sin t)\mathbf{j} + (a \cos t)\mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = |a|;$$

$$-\sqrt{x^2 + z^2} = -\sqrt{0 + a^2 \sin^2 t} = \begin{cases} -|a| \sin t, & 0 \leq t \leq \pi \\ |a| \sin t, & \pi \leq t \leq 2\pi \end{cases} \Rightarrow \int_C f(x, y, z) ds = \int_0^\pi -|a|^2 \sin t dt + \int_\pi^{2\pi} |a|^2 \sin t dt \\ = [a^2 \cos t]_0^\pi - [a^2 \cos t]_\pi^{2\pi} = [a^2(-1) - a^2] - [a^2 - a^2(-1)] = -4a^2$$

$$19. \mathbf{r}(x) = xi + yj = xi + \frac{x^2}{2}j, 0 \leq x \leq 2 \Rightarrow \frac{d\mathbf{r}}{dx} = \mathbf{i} + x\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dx} \right| = \sqrt{1+x^2}; f(x, y) = f\left(x, \frac{x^2}{2}\right) = \frac{x^3}{\left(\frac{x^2}{2}\right)} = 2x \Rightarrow \int_C f ds \\ = \int_0^2 (2x) \sqrt{1+x^2} dx = \left[\frac{2}{3}(1+x^2)^{3/2} \right]_0^2 = \frac{2}{3}(5^{3/2} - 1) = \frac{10\sqrt{5} - 2}{3}$$

$$20. \mathbf{r}(x) = xi + \frac{x^2}{2}j \Rightarrow \frac{d\mathbf{r}}{dx} = \mathbf{i} + x\mathbf{j}, 0 \leq x \leq 1 \Rightarrow \left| \frac{d\mathbf{r}}{dx} \right| = \sqrt{1+x^2}; f(x, y) = f\left(x, \frac{x^2}{2}\right) = \frac{x + \left(\frac{x^4}{4}\right)}{\sqrt{1+x^2}} = \frac{4x+x^4}{4\sqrt{1+x^2}} \\ \Rightarrow \int_C f ds = \int_0^1 \left(\frac{4x+x^4}{4\sqrt{1+x^2}} \right) \sqrt{1+x^2} dx = \int_0^1 \left(x + \frac{x^4}{4} \right) dx = \left[\frac{x^2}{2} + \frac{x^5}{20} \right]_0^1 = \frac{1}{2} + \frac{1}{20} = \frac{11}{20}$$

$$21. \mathbf{r}(t) = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j}, 0 \leq t \leq \frac{\pi}{2} \Rightarrow \frac{d\mathbf{r}}{dt} = (-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 2; f(x, y) = f(2 \cos t, 2 \sin t) \\ = 2 \cos t + 2 \sin t \Rightarrow \int_C f ds = \int_0^{\pi/2} (2 \cos t + 2 \sin t)(2) dt = [4 \sin t - 4 \cos t]_0^{\pi/2} = 4 - (-4) = 8$$

$$22. \mathbf{r}(t) = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j}, \frac{\pi}{2} \geq t \geq \frac{\pi}{4} \Rightarrow \frac{d\mathbf{r}}{dt} = (-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 2; f(x, y) = f(2 \cos t, 2 \sin t) \\ = 4 \cos^2 t - 2 \sin t \Rightarrow \int_C f ds = \int_{\pi/2}^{\pi/4} (4 \cos^2 t - 2 \sin t)(2) dt = [4t + 2 \sin 2t + 4 \cos t]_{\pi/2}^{\pi/4} \\ = (\pi + 2 + 2\sqrt{2}) - (2\pi + 0 + 0) = 2(1 + \sqrt{2}) - \pi$$

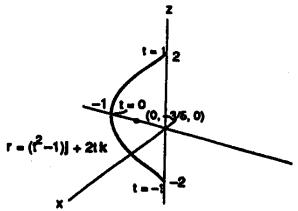
$$23. \mathbf{r}(t) = (t^2 - 1)\mathbf{j} + 2t\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 2t\mathbf{j} + 2\mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 2\sqrt{t^2 + 1}; M = \int_C \delta(x, y, z) ds = \int_0^1 \delta(t, y, z) dt \\ = \int_0^1 \left(\frac{3}{2}t \right) (2\sqrt{t^2 + 1}) dt = [(t^2 + 1)]_0^1 = 2^{3/2} - 1 = 2\sqrt{2} - 1$$

24. $\mathbf{r}(t) = (t^2 - 1)\mathbf{j} + 2t\mathbf{k}$, $-1 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 2t\mathbf{j} + 2\mathbf{k}$

$$\Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = 2\sqrt{t^2 + 1}; M = \int_C \delta(x, y, z) ds$$

$$= \int_{-1}^1 (15\sqrt{(t^2 - 1) + 2})(2\sqrt{t^2 + 1}) dt$$

$$= \int_{-1}^1 30(t^2 + 1) dt = \left[30\left(\frac{t^3}{3} + t\right) \right]_{-1}^1 = 60\left(\frac{1}{3} + 1\right) = 80;$$



$$M_{xz} = \int_C y \delta(x, y, z) ds = \int_{-1}^1 (t^2 - 1)[30(t^2 + 1)] dt = \int_{-1}^1 30(t^4 - 1) dt = \left[30\left(\frac{t^5}{5} - t\right) \right]_{-1}^1 = 60\left(\frac{1}{5} - 1\right) = -48$$

$$\Rightarrow \bar{y} = \frac{M_{xz}}{M} = -\frac{48}{80} = -\frac{3}{5}; M_{yz} = \int_C x \delta(x, y, z) ds = \int_C 0 \delta ds = 0 \Rightarrow \bar{x} = 0; \bar{z} = 0 \text{ by symmetry (since } \delta \text{ is independent of } z\text{)} \Rightarrow (\bar{x}, \bar{y}, \bar{z}) = \left(0, -\frac{3}{5}, 0\right)$$

25. $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + \sqrt{2}t\mathbf{j} + (4 - t^2)\mathbf{k}$, $0 \leq t \leq 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} - 2t\mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{2 + 2 + 4t^2} = 2\sqrt{1 + t^2}$

$$(a) M = \int_C \delta ds = \int_0^1 (3t)(2\sqrt{1+t^2}) dt = \left[2(1+t^2)^{3/2} \right]_0^1 = 2(2^{3/2} - 1) = 4\sqrt{2} - 2$$

$$(b) M = \int_C \delta ds = \int_0^1 (1)(2\sqrt{1+t^2}) dt = [t\sqrt{1+t^2} + \ln(t + \sqrt{1+t^2})]_0^1 = [\sqrt{2} + \ln(1 + \sqrt{2})] - (0 + \ln 1) = \sqrt{2} + \ln(1 + \sqrt{2})$$

26. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + \frac{2}{3}t^{3/2}\mathbf{k}$, $0 \leq t \leq 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2\mathbf{j} + t^{1/2}\mathbf{k} \Rightarrow \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{1+4+t} = \sqrt{5+t}$

$$M = \int_C \delta ds = \int_0^2 (3\sqrt{5+t})(\sqrt{5+t}) dt = \int_0^2 3(5+t) dt = \left[\frac{3}{2}(5+t)^2 \right]_0^2 = \frac{3}{2}(7^2 - 5^2) = \frac{3}{2}(24) = 36;$$

$$M_{yz} = \int_C x \delta ds = \int_0^2 t[3(5+t)] dt = \int_0^2 (15t + 3t^2) dt = \left[\frac{15}{2}t^2 + t^3 \right]_0^2 = 30 + 8 = 38;$$

$$M_{xz} = \int_C y \delta ds = \int_0^2 2t[3(5+t)] dt = 2 \int_0^2 (15t + 3t^2) dt = 76; M_{xy} = \int_C z \delta ds = \int_0^2 \frac{2}{3}t^{3/2}[3(5+t)] dt = \int_0^2 (10t^{3/2} + 2t^{5/2}) dt = \left[4t^{5/2} + \frac{4}{7}t^{7/2} \right]_0^2 = 4(2)^{5/2} + \frac{4}{7}(2)^{7/2} = 16\sqrt{2} + \frac{32}{7}\sqrt{2} = \frac{144}{7}\sqrt{2} \Rightarrow \bar{x} = \frac{M_{yz}}{M}$$

$$= \frac{38}{36} = \frac{19}{18}, \bar{y} = \frac{M_{xz}}{M} = \frac{76}{36} = \frac{19}{9}, \text{ and } \bar{z} = \frac{M_{xy}}{M} = \frac{144\sqrt{2}}{7 \cdot 36} = \frac{4}{7}\sqrt{2}$$

27. Let $x = a \cos t$ and $y = a \sin t$, $0 \leq t \leq 2\pi$. Then $\frac{dx}{dt} = -a \sin t$, $\frac{dy}{dt} = a \cos t$, $\frac{dz}{dt} = 0$

$$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = a dt; I_z = \int_C (x^2 + y^2) \delta ds = \int_0^{2\pi} (a^2 \sin^2 t + a^2 \cos^2 t) a \delta dt \\ = \int_0^{2\pi} a^3 \delta dt = 2\pi a^3; M = \int_C \delta(x, y, z) ds = \int_0^{2\pi} \delta a dt = 2\pi a \delta \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{2\pi a^3 \delta}{2\pi a \delta}} = a.$$

28. $r(t) = tj + (2 - 2t)k$, $0 \leq t \leq 1 \Rightarrow \frac{dr}{dt} = j - 2k \Rightarrow \left|\frac{dr}{dt}\right| = \sqrt{5}$; $M = \int_C \delta ds = \int_0^1 \delta \sqrt{5} dt = \delta \sqrt{5}$;

$$I_x = \int_C (y^2 + z^2) \delta ds = \int_0^1 [t^2 + (2 - 2t)^2] \delta \sqrt{5} dt = \int_0^1 (5t^2 - 8t + 4) \delta \sqrt{5} dt = \delta \sqrt{5} \left[\frac{5}{3}t^3 - 4t^2 + 4t \right]_0^1 = \frac{5}{3} \delta \sqrt{5};$$

$$I_y = \int_C (x^2 + z^2) \delta ds = \int_0^1 [0^2 + (2 - 2t)^2] \delta \sqrt{5} dt = \int_0^1 (4t^2 - 8t + 4) \delta \sqrt{5} dt = \delta \sqrt{5} \left[\frac{4}{3}t^3 - 4t^2 + 4t \right]_0^1 = \frac{4}{3} \delta \sqrt{5};$$

$$I_z = \int_C (x^2 + y^2) \delta ds = \int_0^1 (0^2 + t^2) \delta \sqrt{5} dt = \delta \sqrt{5} \left[\frac{t^3}{3} \right]_0^1 = \frac{1}{3} \delta \sqrt{5} \Rightarrow R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{5}{3}}, R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}},$$

$$\text{and } R_z = \sqrt{\frac{I_z}{M}} = \frac{1}{\sqrt{3}}$$

29. $r(t) = (\cos t)i + (\sin t)j + tk$, $0 \leq t \leq 2\pi \Rightarrow \frac{dr}{dt} = (-\sin t)i + (\cos t)j + k \Rightarrow \left|\frac{dr}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$;

$$(a) M = \int_C \delta ds = \int_0^{2\pi} \delta \sqrt{2} dt = 2\pi \delta \sqrt{2}; I_z = \int_C (x^2 + y^2) \delta ds = \int_0^{2\pi} (\cos^2 t + \sin^2 t) \delta \sqrt{2} dt = 2\pi \delta \sqrt{2}$$

$$\Rightarrow R_z = \sqrt{\frac{I_z}{M}} = 1$$

$$(b) M = \int_C \delta(x, y, z) ds = \int_0^{4\pi} \delta \sqrt{2} dt = 4\pi \delta \sqrt{2} \text{ and } I_z = \int_C (x^2 + y^2) \delta ds = \int_0^{4\pi} \delta \sqrt{2} dt = 4\pi \delta \sqrt{2}$$

$$\Rightarrow R_z = \sqrt{\frac{I_z}{M}} = 1$$

30. $r(t) = (t \cos t)i + (t \sin t)j + \frac{2\sqrt{2}}{3} t^{3/2} k$, $0 \leq t \leq 1 \Rightarrow \frac{dr}{dt} = (\cos t - t \sin t)i + (\sin t + t \cos t)j + \sqrt{2}k$

$$\Rightarrow \left|\frac{dr}{dt}\right| = \sqrt{(t+1)^2} = t+1 \text{ for } 0 \leq t \leq 1; M = \int_C \delta ds = \int_0^1 (t+1) dt = \left[\frac{1}{2}(t+1)^2 \right]_0^1 = \frac{1}{2}(2^2 - 1^2) = \frac{3}{2};$$

$$M_{xy} = \int_C z \delta ds = \int_0^1 \left(\frac{2\sqrt{2}}{3} t^{3/2} \right) (t+1) dt = \frac{2\sqrt{2}}{3} \int_0^1 (t^{5/2} + t^{3/2}) dt = \frac{2\sqrt{2}}{3} \left[\frac{2}{7} t^{7/2} + \frac{2}{5} t^{5/2} \right]_0^1$$

$$\begin{aligned}
&= \frac{2\sqrt{2}}{3} \left(\frac{2}{7} + \frac{2}{5} \right) = \frac{2\sqrt{2}}{3} \left(\frac{24}{35} \right) = \frac{16\sqrt{2}}{35} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{16\sqrt{2}}{35} \right) \left(\frac{2}{3} \right) = \frac{32\sqrt{2}}{105}; I_z = \int_C (x^2 + y^2) \delta \, ds \\
&= \int_0^1 (t^2 \cos^2 t + t^2 \sin^2 t)(t+1) \, dt = \int_0^1 (t^3 + t^2) \, dt = \left[\frac{t^4}{4} + \frac{t^3}{3} \right]_0^1 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{7}{18}}
\end{aligned}$$

31. $\delta(x, y, z) = 2 - z$ and $r(t) = (\cos t)\mathbf{j} + (\sin t)\mathbf{k}$, $0 \leq t \leq \pi \Rightarrow M = 2\pi - 2$ as found in Example 4 of the text;

$$\begin{aligned}
\text{also } \left| \frac{dr}{dt} \right| &= 1; I_x = \int_C (y^2 + z^2) \delta \, ds = \int_0^\pi (\cos^2 t + \sin^2 t)(2 - \sin t) \, dt = \int_0^\pi (2 - \sin t) \, dt = 2\pi - 2 \Rightarrow R_x = \sqrt{\frac{I_x}{M}} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
32. \quad r(t) &= ti + \frac{2\sqrt{2}}{3} t^{3/2} \mathbf{j} + \frac{t^2}{2} \mathbf{k}, \quad 0 \leq t \leq 2 \Rightarrow \frac{dr}{dt} = i + \sqrt{2} t^{1/2} \mathbf{j} + t \mathbf{k} \Rightarrow \left| \frac{dr}{dt} \right| = \sqrt{1 + 2t + t^2} = \sqrt{(1+t)^2} = 1+t \text{ for} \\
&0 \leq t \leq 2; M = \int_C \delta \, ds = \int_0^2 \left(\frac{1}{t+1} \right) (1+t) \, dt = \int_0^2 dt = 2; M_{yz} = \int_C x \delta \, ds = \int_0^2 t \left(\frac{1}{t+1} \right) (1+t) \, dt = \left[\frac{t^2}{2} \right]_0^2 = 2; \\
&M_{xz} = \int_C y \delta \, ds = \int_0^2 \frac{2\sqrt{2}}{3} t^{3/2} \, dt = \left[\frac{4\sqrt{2}}{15} t^{5/2} \right]_0^2 = \frac{32}{15}; M_{xy} = \int_C z \delta \, ds = \int_0^2 \frac{t^2}{2} \, dt = \left[\frac{t^3}{6} \right]_0^2 = \frac{8}{3} \Rightarrow \bar{x} = \frac{M_{yz}}{M} = 1, \\
&\bar{y} = \frac{M_{xz}}{M} = \frac{16}{15}, \text{ and } \bar{z} = \frac{M_{xy}}{M} = \frac{4}{3}; I_x = \int_C (y^2 + z^2) \delta \, ds = \int_0^2 \left(\frac{8}{9} t^3 + \frac{1}{4} t^4 \right) \, dt = \left[\frac{2}{9} t^4 + \frac{t^5}{20} \right]_0^2 = \frac{32}{9} + \frac{32}{20} = \frac{232}{45}, \\
&I_y = \int_C (x^2 + z^2) \delta \, ds = \int_0^2 \left(t^2 + \frac{1}{4} t^4 \right) \, dt = \left[\frac{t^3}{3} + \frac{t^5}{20} \right]_0^2 = \frac{8}{3} + \frac{32}{20} = \frac{64}{15}; I_z = \int_C (x^2 + y^2) \delta \, ds \\
&= \int_0^2 \left(t^2 + \frac{8}{9} t^3 \right) \, dt = \left[\frac{t^3}{3} + \frac{2}{9} t^4 \right]_0^2 = \frac{8}{3} + \frac{32}{9} = \frac{56}{9} \Rightarrow R_x = \sqrt{\frac{I_x}{M}} = \frac{2}{3} \sqrt{\frac{29}{5}}, R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{32}{15}}, \text{ and} \\
&R_z = \sqrt{\frac{I_z}{M}} = \frac{2}{3} \sqrt{7}
\end{aligned}$$

33-36. Example CAS commands:

Maple:

```

x:= t -> cos(2*t); y:= t -> sin(2*t);
z:= t -> t^(5/2);
f:=(x,y,z)->(1+(9/4)*z^(1/3))^(1/4);
sqrt(D(x)(t)^2 + D(y)(t)^2 + D(z)(t)^2): absvee := unapply(% ,t);
a:= 0; b:= 2*Pi;
integrand:= simplify(f(x(t),y(t),z(t))*absvee(t));
int(integrand,t=a..b);
evalf(%);

```

Mathematica:

```

Clear[x,y,z,t]
r[t]= {x[t],y[t],z[t]}
f[x_,y_,z_]=(1+ 9/4 z^(1/3))^(1/4)
x[t_]= Cos[2 t]
y[t_]= Sin[2 t]

```

```

z[t_] = t^(5/2)
{a,b} = {0,2Pi};
v[t_] = r'[t]
s[t_] = Sqrt[ v[t] . v[t] ]
integrand = f[x[t],y[t],z[t]] s[t]
NIntegrate[ integrand, {t,a,b} ]

```

13.2 VECTOR FIELDS, WORK, CIRCULATION, AND FLUX

1. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2} \Rightarrow \frac{\partial f}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x) = -x(x^2 + y^2 + z^2)^{-3/2}$; similarly,

$$\frac{\partial f}{\partial y} = -y(x^2 + y^2 + z^2)^{-3/2} \text{ and } \frac{\partial f}{\partial z} = -z(x^2 + y^2 + z^2)^{-3/2} \Rightarrow \nabla f = \frac{-xi - yj - zk}{(x^2 + y^2 + z^2)^{3/2}}$$

2. $f(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2} = \frac{1}{2} \ln(x^2 + y^2 + z^2) \Rightarrow \frac{\partial f}{\partial x} = \frac{1}{2} \left(\frac{1}{x^2 + y^2 + z^2} \right)(2x) = \frac{x}{x^2 + y^2 + z^2}$;

similarly, $\frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2 + z^2}$ and $\frac{\partial f}{\partial z} = \frac{z}{x^2 + y^2 + z^2} \Rightarrow \nabla f = \frac{xi + yj + zk}{x^2 + y^2 + z^2}$

3. $g(x, y, z) = e^x - \ln(x^2 + y^2) \Rightarrow \frac{\partial g}{\partial x} = -\frac{2x}{x^2 + y^2}, \frac{\partial g}{\partial y} = -\frac{2y}{x^2 + y^2} \text{ and } \frac{\partial g}{\partial z} = e^z$

$$\Rightarrow \nabla g = \left(\frac{-2x}{x^2 + y^2} \right) i - \left(\frac{-2y}{x^2 + y^2} \right) j + e^z k$$

4. $g(x, y, z) = xy + yz + xz \Rightarrow \frac{\partial g}{\partial x} = y + z, \frac{\partial g}{\partial y} = x + z, \text{ and } \frac{\partial g}{\partial z} = y + x \Rightarrow \nabla g = (y + z)i + (x + z)j + (x + y)k$

5. $|\mathbf{F}|$ inversely proportional to the square of the distance from (x, y) to the origin $\Rightarrow \sqrt{(M(x, y))^2 + (N(x, y))^2}$

$$= \frac{k}{x^2 + y^2}, k > 0; \mathbf{F} \text{ points toward the origin} \Rightarrow \mathbf{F} \text{ is in the direction of } \mathbf{n} = \frac{-x}{\sqrt{x^2 + y^2}} i - \frac{y}{\sqrt{x^2 + y^2}} j$$

$$\Rightarrow \mathbf{F} = a\mathbf{n}, \text{ for some constant } a > 0. \text{ Then } M(x, y) = \frac{-ax}{\sqrt{x^2 + y^2}} \text{ and } N(x, y) = \frac{-ay}{\sqrt{x^2 + y^2}}$$

$$\Rightarrow \sqrt{(M(x, y))^2 + (N(x, y))^2} = a \Rightarrow a = \frac{k}{x^2 + y^2} \Rightarrow \mathbf{F} = \frac{-kx}{(x^2 + y^2)^{3/2}} i - \frac{ky}{(x^2 + y^2)^{3/2}} j, \text{ for any constant } k > 0$$

6. Given $x^2 + y^2 = a^2 + b^2$, let $x = \sqrt{a^2 + b^2} \cos t$ and $y = -\sqrt{a^2 + b^2} \sin t$. Then

$$\mathbf{r} = (\sqrt{a^2 + b^2} \cos t)i - (\sqrt{a^2 + b^2} \sin t)j \text{ traces the circle in a clockwise direction as } t \text{ goes from } 0 \text{ to } 2\pi$$

$\Rightarrow \mathbf{v} = (-\sqrt{a^2 + b^2} \sin t)i - (\sqrt{a^2 + b^2} \cos t)j$ is tangent to the circle in a clockwise direction. Thus, let

$$\mathbf{F} = \mathbf{v} \Rightarrow \mathbf{F} = yi - xj \text{ and } \mathbf{F}(0, 0) = \mathbf{0}.$$

7. Substitute the parametric representations for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field \mathbf{F} , and calculate the work $W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

$$(a) \mathbf{F} = 3t\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 9t \Rightarrow W = \int_0^1 9t \, dt = \frac{9}{2}$$

$$(b) \mathbf{F} = 3t^2\mathbf{i} + 2t\mathbf{j} + 4t^4\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 7t^2 + 16t^7 \Rightarrow W = \int_0^1 (7t^2 + 16t^7) \, dt = \left[\frac{7}{3}t^3 + 2t^8 \right]_0^1 = \frac{7}{3} + 2 = \frac{13}{3}$$

$$(c) \mathbf{r}_1 = t\mathbf{i} + t\mathbf{j} \text{ and } \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}; \mathbf{F}_1 = 3t\mathbf{i} + 2t\mathbf{j} \text{ and } \frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 5t \Rightarrow W_1 = \int_0^1 5t \, dt = \frac{5}{2};$$

$$\mathbf{F}_2 = 3\mathbf{i} + 2\mathbf{j} + 4t\mathbf{k} \text{ and } \frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 4t \Rightarrow W_2 = \int_0^1 4t \, dt = 2 \Rightarrow W = W_1 + W_2 = \frac{9}{2}$$

8. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field \mathbf{F} , and calculate the work $W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

$$(a) \mathbf{F} = \left(\frac{1}{t^2+1} \right) \mathbf{j} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{1}{t^2+1} \Rightarrow W = \int_0^1 \frac{1}{t^2+1} \, dt = [\tan^{-1} t]_0^1 = \frac{\pi}{4}$$

$$(b) \mathbf{F} = \left(\frac{1}{t^2+1} \right) \mathbf{j} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{2t}{t^2+1} \Rightarrow W = \int_0^1 \frac{2t}{t^2+1} \, dt = [\ln(t^2+1)]_0^1 = \ln 2$$

$$(c) \mathbf{r}_1 = t\mathbf{i} + t\mathbf{j} \text{ and } \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}; \mathbf{F}_1 = \left(\frac{1}{t^2+1} \right) \mathbf{j} \text{ and } \frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = \frac{1}{t^2+1}; \mathbf{F}_2 = \frac{1}{2}\mathbf{j} \text{ and } \frac{d\mathbf{r}_2}{dt} = \mathbf{k}$$

$$\Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0 \Rightarrow W = \int_0^1 \frac{1}{t^2+1} \, dt = \frac{\pi}{4}$$

9. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field \mathbf{F} , and calculate the work $W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

$$(a) \mathbf{F} = \sqrt{t}\mathbf{i} - 2t\mathbf{j} + \sqrt{t}\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2\sqrt{t} - 2t \Rightarrow W = \int_0^1 (2\sqrt{t} - 2t) \, dt = \left[\frac{4}{3}t^{3/2} - t^2 \right]_0^1 = \frac{1}{3}$$

$$(b) \mathbf{F} = t^2\mathbf{i} - 2t\mathbf{j} + t\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 4t^4 - 3t^2 \Rightarrow W = \int_0^1 (4t^4 - 3t^2) \, dt = \left[\frac{4}{5}t^5 - t^3 \right]_0^1 = -\frac{1}{5}$$

$$(c) \quad \mathbf{r}_1 = t\mathbf{i} + t\mathbf{j} \text{ and } \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}; \quad \mathbf{F}_1 = -2t\mathbf{j} + \sqrt{t}\mathbf{k} \text{ and } \frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = -2t \Rightarrow W_1 = \int_0^1 -2t \, dt \\ = -1; \quad \mathbf{F}_2 = \sqrt{t}\mathbf{i} - 2\mathbf{j} + \mathbf{k} \text{ and } \frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 1 \Rightarrow W_2 = \int_0^1 dt = 1 \Rightarrow W = W_1 + W_2 = 0$$

10. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field \mathbf{F} , and calculate the work $W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

$$(a) \quad \mathbf{F} = t^2\mathbf{i} + t^2\mathbf{j} + t^2\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 3t^2 \Rightarrow W = \int_0^1 3t^2 \, dt = 1$$

$$(b) \quad \mathbf{F} = t^3\mathbf{i} - t^6\mathbf{j} + t^5\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 + 2t^7 + 4t^8 \Rightarrow W = \int_0^1 (t^3 + 2t^7 + 4t^8) \, dt \\ = \left[\frac{t^4}{4} + \frac{t^8}{4} + \frac{4}{9}t^9 \right]_0^1 = \frac{17}{18}$$

$$(c) \quad \mathbf{r}_1 = t\mathbf{i} + t\mathbf{j} \text{ and } \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}; \quad \mathbf{F}_1 = t^2\mathbf{i} \text{ and } \frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = t^2 \Rightarrow W_1 = \int_0^1 t^2 \, dt = \frac{1}{3}; \\ \mathbf{F}_2 = \mathbf{i} + t\mathbf{j} + t\mathbf{k} \text{ and } \frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t \Rightarrow W_2 = \int_0^1 t \, dt = \frac{1}{2} \Rightarrow W = W_1 + W_2 = \frac{5}{6}$$

11. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field \mathbf{F} , and calculate the work $W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

$$(a) \quad \mathbf{F} = (3t^2 - 3t)\mathbf{i} + 3t\mathbf{j} + \mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 3t^2 + 1 \Rightarrow W = \int_0^1 (3t^2 + 1) \, dt = [t^3 + t]_0^1 = 2$$

$$(b) \quad \mathbf{F} = (3t^2 - 3t)\mathbf{i} + 3t^4\mathbf{j} + \mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t^5 + 4t^3 + 3t^2 - 3t \\ \Rightarrow W = \int_0^1 (6t^5 + 4t^3 + 3t^2 - 3t) \, dt = \left[t^6 + t^4 + t^3 - \frac{3}{2}t^2 \right]_0^1 = \frac{3}{2}$$

$$(c) \quad \mathbf{r}_1 = t\mathbf{i} + t\mathbf{j} \text{ and } \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}; \quad \mathbf{F}_1 = (3t^2 - 3t)\mathbf{i} + \mathbf{k} \text{ and } \frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 3t^2 - 3t \\ \Rightarrow W_1 = \int_0^1 (3t^2 - 3t) \, dt = \left[t^3 - \frac{3}{2}t^2 \right]_0^1 = -\frac{1}{2}; \quad \mathbf{F}_2 = 3t\mathbf{j} + \mathbf{k} \text{ and } \frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 1 \Rightarrow W_2 = \int_0^1 dt = 1 \\ \Rightarrow W = W_1 + W_2 = \frac{1}{2}$$

12. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field \mathbf{F} , and calculate the work $W = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

$$(a) \mathbf{F} = 2t\mathbf{i} + 2t\mathbf{j} + 2t\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t \Rightarrow W = \int_0^1 6t \, dt = [3t^2]_0^1 = 3$$

$$(b) \mathbf{F} = (t^2 + t^4)\mathbf{i} + (t^4 + t)\mathbf{j} + (t + t^2)\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t^5 + 5t^4 + 3t^2$$

$$\Rightarrow W = \int_0^1 (6t^5 + 5t^4 + 3t^2) \, dt = [t^6 + t^5 + t^3]_0^1 = 3$$

$$(c) \mathbf{r}_1 = t\mathbf{i} + t\mathbf{j} \text{ and } \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}; \mathbf{F}_1 = t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k} \text{ and } \frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 2t$$

$$\Rightarrow W_1 = \int_0^1 2t \, dt = 1; \mathbf{F}_2 = (1+t)\mathbf{i} + (t+1)\mathbf{j} + 2\mathbf{k} \text{ and } \frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 2 \Rightarrow W_2 = \int_0^1 2 \, dt = 2$$

$$\Rightarrow W = W_1 + W_2 = 3$$

$$13. \mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 1, \text{ and } \mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k} \Rightarrow \mathbf{F} = t^3\mathbf{i} + t^2\mathbf{j} - t^3\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$$

$$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t^3 \Rightarrow \text{work} = \int_0^1 2t^3 \, dt = \frac{1}{2}$$

$$14. \mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \frac{t}{6}\mathbf{k}, 0 \leq t \leq 2\pi, \text{ and } \mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x+y)\mathbf{k}$$

$$\Rightarrow \mathbf{F} = (2 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + (\cos t + \sin t)\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \frac{1}{6}\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$$

$$= 5 \cos^2 t - 2 + \frac{1}{6} \cos t + \frac{1}{6} \sin t \Rightarrow \text{work} = \int_0^{2\pi} \left(5 \cos^2 t - 2 + \frac{1}{6} \cos t + \frac{1}{6} \sin t \right) dt$$

$$= \left[\frac{5}{2}t + \frac{5}{4} \sin 2t - 2t + \frac{1}{6} \sin t - \frac{1}{6} \cos t \right]_0^{2\pi} = 5\pi - 4\pi = \pi$$

$$15. \mathbf{r} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 2\pi, \text{ and } \mathbf{F} = zi + xj + yk \Rightarrow \mathbf{F} = ti + (\sin t)\mathbf{j} + (\cos t)\mathbf{k} \text{ and}$$

$$\frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t \cos t - \sin^2 t + \cos t \Rightarrow \text{work} = \int_0^{2\pi} (t \cos t - \sin^2 t + \cos t) dt$$

$$= \left[\cos t + t \sin t - \frac{t}{2} + \frac{\sin 2t}{4} + \sin t \right]_0^{2\pi} = -\pi$$

$$16. \mathbf{r} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \frac{t}{6}\mathbf{k}, 0 \leq t \leq 2\pi, \text{ and } \mathbf{F} = 6zi + y^2\mathbf{j} + 12x\mathbf{k} \Rightarrow \mathbf{F} = ti + (\cos^2 t)\mathbf{j} + (12 \sin t)\mathbf{k} \text{ and}$$

$$\frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \frac{1}{6}\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t \cos t - \sin t \cos^2 t + 2 \sin t$$

$$\Rightarrow \text{work} = \int_0^{2\pi} (t \cos t - \sin t \cos^2 t + 2 \sin t) dt = [\cos t + t \sin t + \frac{1}{3} \cos^3 t - 2 \cos t]_0^{2\pi} = 0$$

17. $x = t$ and $y = x^2 = t^2 \Rightarrow r = ti + t^2j$, $-1 \leq t \leq 2$, and $\mathbf{F} = xyi + (x+y)j \Rightarrow \mathbf{F} = t^3i + (t+t^2)j$ and

$$\begin{aligned} \frac{dr}{dt} &= i + 2tj \Rightarrow \mathbf{F} \cdot \frac{dr}{dt} = t^3 + (2t^2 + 2t^3) = 3t^3 + 2t^2 \Rightarrow \int_C xy \, dx + (x+y) \, dy = \int_C \mathbf{F} \cdot \frac{dr}{dt} \, dt = \int_{-1}^2 (3t^3 + 2t^2) \, dt \\ &= \left[\frac{3}{4}t^4 + \frac{2}{3}t^3 \right]_{-1}^2 = \left(12 + \frac{16}{3} \right) - \left(\frac{3}{4} - \frac{2}{3} \right) = \frac{45}{4} + \frac{18}{3} = \frac{69}{4} \end{aligned}$$

18. Along $(0,0)$ to $(1,0)$: $r = ti$, $0 \leq t \leq 1$, and $\mathbf{F} = (x-y)i + (x+y)j \Rightarrow \mathbf{F} = ti + tj$ and $\frac{dr}{dt} = i \Rightarrow \mathbf{F} \cdot \frac{dr}{dt} = t$;

Along $(1,0)$ to $(0,1)$: $r = (1-t)i + tj$, $0 \leq t \leq 1$, and $\mathbf{F} = (x-y)i + (x+y)j \Rightarrow \mathbf{F} = (1-2t)i + j$ and

$$\frac{dr}{dt} = -i + j \Rightarrow \mathbf{F} \cdot \frac{dr}{dt} = 2t;$$

Along $(0,1)$ to $(0,0)$: $r = (1-t)j$, $0 \leq t \leq 1$, and $\mathbf{F} = (x-y)i + (x+y)j \Rightarrow \mathbf{F} = (t-1)i + (1-t)j$ and

$$\begin{aligned} \frac{dr}{dt} &= -j \Rightarrow \mathbf{F} \cdot \frac{dr}{dt} = t-1 \Rightarrow \int_C (x-y) \, dx + (x+y) \, dy = \int_0^1 t \, dt + \int_0^1 2t \, dt + \int_0^1 (t-1) \, dt = \int_0^1 (4t-1) \, dt \\ &= [2t^2 - t]_0^1 = 2 - 1 = 1 \end{aligned}$$

19. $r = xi + yj = y^2i + yj$, $2 \geq y \geq -1$, and $\mathbf{F} = x^2i - yj = y^4i - yj \Rightarrow \frac{dr}{dy} = 2yi + j$ and $\mathbf{F} \cdot \frac{dr}{dy} = 2y^5 - y$

$$\Rightarrow \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_2^{-1} \mathbf{F} \cdot \frac{dr}{dy} \, dy = \int_2^{-1} (2y^5 - y) \, dy = \left[\frac{1}{3}y^6 - \frac{1}{2}y^2 \right]_2^{-1} = \left(\frac{1}{3} - \frac{1}{2} \right) - \left(\frac{64}{3} - \frac{4}{2} \right) = \frac{3}{2} - \frac{63}{3} = -\frac{39}{2}$$

20. $r = (\cos t)i + (\sin t)j$, $0 \leq t \leq \frac{\pi}{2}$, and $\mathbf{F} = yi - xj \Rightarrow \mathbf{F} = (\sin t)i - (\cos t)j$ and $\frac{dr}{dt} = (-\sin t)i + (\cos t)j$

$$\Rightarrow \mathbf{F} \cdot \frac{dr}{dt} = -\sin^2 t - \cos^2 t = -1 \Rightarrow \int_C \mathbf{F} \cdot dr = \int_0^{\pi/2} (-1) \, dt = -\frac{\pi}{2}$$

21. $r = (i+j) + t(i+2j) = (1+t)i + (1+2t)j$, $0 \leq t \leq 1$, and $\mathbf{F} = xyi + (y-x)j \Rightarrow \mathbf{F} = (1+3t+2t^2)i + tj$ and

$$\frac{dr}{dt} = i + 2j \Rightarrow \mathbf{F} \cdot \frac{dr}{dt} = 1 + 5t + 2t^2 \Rightarrow \text{work} = \int_C \mathbf{F} \cdot \frac{dr}{dt} \, dt = \int_0^1 (1 + 5t + 2t^2) \, dt = \left[t + \frac{5}{2}t^2 + \frac{2}{3}t^3 \right]_0^1 = \frac{25}{6}$$

22. $r = (2 \cos t)i + (2 \sin t)j$, $0 \leq t \leq 2\pi$, and $\mathbf{F} = \nabla f = 2(x+y)i + 2(x+y)j$

$$\Rightarrow \mathbf{F} = 4(\cos t + \sin t)i + 4(\cos t + \sin t)j \text{ and } \frac{dr}{dt} = (-2 \sin t)i + (2 \cos t)j \Rightarrow \mathbf{F} \cdot \frac{dr}{dt}$$

$$= -8(\sin t \cos t + \sin^2 t) + 8(\cos^2 t + \cos t \sin t) = 8(\cos^2 t - \sin^2 t) = 8 \cos 2t \Rightarrow \text{work} = \int_C \nabla f \cdot dr$$

$$= \int_C \mathbf{F} \cdot \frac{dr}{dt} \, dt = \int_0^{2\pi} 8 \cos 2t \, dt = [4 \sin 2t]_0^{2\pi} = 0$$

23. (a) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$, $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}$, and $\mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$,

$$\mathbf{F}_1 = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \text{ and } \mathbf{F}_2 = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}}{dt} = 0 \text{ and } \mathbf{F}_2 \cdot \frac{d\mathbf{r}}{dt} = \sin^2 t + \cos^2 t = 1$$

$$\Rightarrow \text{Circ}_1 = \int_0^{2\pi} 0 dt = 0 \text{ and } \text{Circ}_2 = \int_0^{2\pi} dt = 2\pi; \mathbf{n} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \mathbf{n} = \cos^2 t + \sin^2 t = 1 \text{ and}$$

$$\mathbf{F}_2 \cdot \mathbf{n} = 0 \Rightarrow \text{Flux}_1 = \int_0^{2\pi} dt = 2\pi \text{ and } \text{Flux}_2 = \int_0^{2\pi} 0 dt = 0$$

(b) $\mathbf{r} = (\cos t)\mathbf{i} + (4 \sin t)\mathbf{j}$, $0 \leq t \leq 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (4 \cos t)\mathbf{j}$, $\mathbf{F}_1 = (\cos t)\mathbf{i} + (4 \sin t)\mathbf{j}$, and

$$\mathbf{F}_2 = (-4 \sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}}{dt} = 15 \sin t \cos t \text{ and } \mathbf{F}_2 \cdot \frac{d\mathbf{r}}{dt} = 4 \Rightarrow \text{Circ}_1 = \int_0^{2\pi} 15 \sin t \cos t dt$$

$$= \left[\frac{15}{2} \sin^2 t \right]_0^{2\pi} = 0 \text{ and } \text{Circ}_2 = \int_0^{2\pi} 4 dt = 8\pi; \mathbf{n} = \left(\frac{4}{\sqrt{17}} \cos t \right) \mathbf{i} + \left(\frac{1}{\sqrt{17}} \sin t \right) \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \mathbf{n}$$

$$= \frac{4}{\sqrt{17}} \cos^2 t + \frac{4}{\sqrt{17}} \sin^2 t \text{ and } \mathbf{F}_2 \cdot \mathbf{n} = -\frac{15}{\sqrt{17}} \sin t \cos t \Rightarrow \text{Flux}_1 = \int_0^{2\pi} (\mathbf{F}_1 \cdot \mathbf{n}) |\mathbf{v}| dt = \int_0^{2\pi} \left(\frac{4}{\sqrt{17}} \right) \sqrt{17} dt$$

$$= 8\pi \text{ and } \text{Flux}_2 = \int_0^{2\pi} (\mathbf{F}_2 \cdot \mathbf{n}) |\mathbf{v}| dt = \int_0^{2\pi} \left(-\frac{15}{\sqrt{17}} \sin t \cos t \right) \sqrt{17} dt = \left[-\frac{15}{2} \sin^2 t \right]_0^{2\pi} = 0$$

24. $\mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$, $\mathbf{F}_1 = 2xi - 3y\mathbf{j}$, and $\mathbf{F}_2 = 2xi + (x - y)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j}$,

$$\mathbf{F}_1 = (2a \cos t)\mathbf{i} - (3a \sin t)\mathbf{j}, \text{ and } \mathbf{F}_2 = (2a \cos t)\mathbf{i} + (a \cos t - a \sin t)\mathbf{j} \Rightarrow \mathbf{n} |\mathbf{v}| = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j},$$

$$\mathbf{F}_1 \cdot \mathbf{n} |\mathbf{v}| = 2a^2 \cos^2 t - 3a^2 \sin^2 t, \text{ and } \mathbf{F}_2 \cdot \mathbf{n} |\mathbf{v}| = 2a^2 \cos^2 t + a^2 \sin t \cos t - a^2 \sin^2 t$$

$$\Rightarrow \text{Flux}_1 = \int_0^{2\pi} (2a^2 \cos^2 t + 3a^2 \sin^2 t) dt = 2a^2 \left[\frac{t}{2} + \frac{\sin 2t}{4} \right]_0^{2\pi} - 3a^2 \left[\frac{t}{2} - \frac{\sin 2t}{4} \right]_0^{2\pi} = -\pi a^2, \text{ and}$$

$$\text{Flux}_2 = \int_0^{2\pi} (2a^2 \cos^2 t + a^2 \sin t \cos t - a^2 \sin^2 t) dt = 2a^2 \left[\frac{t}{2} + \frac{\sin 2t}{4} \right]_0^{2\pi} + \frac{a^2}{2} [\sin^2 t]_0^{2\pi} - a^2 \left[\frac{t}{2} - \frac{\sin 2t}{4} \right]_0^{2\pi} = \pi a^2$$

25. $\mathbf{F}_1 = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$, $\frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 0 \Rightarrow \text{Circ}_1 = 0$; $M_1 = a \cos t$,

$$N_1 = a \sin t, dx = -a \sin t dt, dy = a \cos t dt \Rightarrow \text{Flux}_1 = \int_C M_1 dy - N_1 dx = \int_0^\pi (a^2 \cos^2 t + a^2 \sin^2 t) dt$$

$$= \int_0^\pi a^2 dt = a^2 \pi;$$

$$\begin{aligned}\mathbf{F}_2 &= t\mathbf{i}, \frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t \Rightarrow \text{Circ}_2 = \int_{-a}^a t dt = 0; M_2 = t, N_2 = 0, dx = dt, dy = 0 \Rightarrow \text{Flux}_2 \\ &= \int_C M_2 dy - N_2 dx = \int_{-a}^a 0 dt = 0; \text{ therefore, } \text{Circ} = \text{Circ}_1 + \text{Circ}_2 = 0 \text{ and } \text{Flux} = \text{Flux}_1 + \text{Flux}_2 = a^2\pi\end{aligned}$$

$$\begin{aligned}26. \quad \mathbf{F}_1 &= (a^2 \cos^2 t)\mathbf{i} + (a^2 \sin^2 t)\mathbf{j}, \frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = -a^3 \sin t \cos^2 t + a^3 \cos t \sin^2 t \\ &\Rightarrow \text{Circ}_1 = \int_0^\pi (-a^3 \sin t \cos^2 t + a^3 \cos t \sin^2 t) dt = -\frac{2a^3}{3}; M_1 = a^2 \cos^2 t, N_1 = a^2 \sin^2 t, dy = a \cos t dt, \\ &dx = -a \sin t dt \Rightarrow \text{Flux}_1 = \int_C M_1 dy - N_1 dx = \int_0^\pi (a^3 \cos^3 t + a^3 \sin^3 t) dt = \frac{4}{3}a^3; \\ \mathbf{F}_2 &= t^2\mathbf{i}, \frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t^2 \Rightarrow \text{Circ}_2 = \int_{-a}^a t^2 dt = \frac{2a^3}{3}; M_2 = t^2, N_2 = 0, dy = 0, dx = dt \\ &\Rightarrow \text{Flux}_2 = \int_C M_2 dy - N_2 dx = 0; \text{ therefore, } \text{Circ} = \text{Circ}_1 + \text{Circ}_2 = 0 \text{ and } \text{Flux} = \text{Flux}_1 + \text{Flux}_2 = \frac{4}{3}a^3\end{aligned}$$

$$\begin{aligned}27. \quad \mathbf{F}_1 &= (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j}, \frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = a^2 \sin^2 t + a^2 \cos^2 t = a^2 \\ &\Rightarrow \text{Circ}_1 = \int_0^\pi a^2 dt = a^2\pi; M_1 = -a \sin t, N_1 = a \cos t, dx = -a \sin t dt, dy = a \cos t dt \\ &\Rightarrow \text{Flux}_1 = \int_C M_1 dy - N_1 dx = \int_0^\pi (-a^2 \sin t \cos t + a^2 \sin t \cos t) dt = 0; \mathbf{F}_2 = t\mathbf{j}, \frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0 \\ &\Rightarrow \text{Circ}_2 = 0; M_2 = 0, N_2 = t, dx = dt, dy = 0 \Rightarrow \text{Flux}_2 = \int_C M_2 dy - N_2 dx = \int_{-a}^a -t dt = 0; \text{ therefore,} \\ &\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = a^2\pi \text{ and } \text{Flux} = \text{Flux}_1 + \text{Flux}_2 = 0\end{aligned}$$

$$\begin{aligned}28. \quad \mathbf{F}_1 &= (-a^2 \sin^2 t)\mathbf{i} + (a^2 \cos^2 t)\mathbf{j}, \frac{d\mathbf{r}_1}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = a^3 \sin^3 t + a^3 \cos^3 t \\ &\Rightarrow \text{Circ}_1 = \int_0^\pi (a^2 \sin^3 t + a^3 \cos^3 t) dt = \frac{4}{3}a^3; M_1 = -a^2 \sin^2 t, N_1 = a^2 \cos^2 t, dy = a \cos t dt, dx = -a \sin t dt \\ &\Rightarrow \text{Flux}_1 = \int_C M_1 dy - N_1 dx = \int_0^\pi (-a^3 \cos t \sin^2 t + a^3 \sin t \cos^2 t) dt = \frac{2}{3}a^3; \mathbf{F}_2 = t^2\mathbf{j}, \frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0 \\ &\Rightarrow \text{Circ}_2 = 0; M_2 = 0, N_2 = t^2, dy = 0, dx = dt \Rightarrow \text{Flux}_2 = \int_C M_2 dy - N_2 dx = \int_{-a}^a -t^2 dt = -\frac{2}{3}a^3; \text{ therefore,} \\ &\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = \frac{4}{3}a^3 \text{ and } \text{Flux} = \text{Flux}_1 + \text{Flux}_2 = 0\end{aligned}$$

29. (a) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \leq t \leq \pi$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$ and

$$\begin{aligned}\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= (\cos t + \sin t)\mathbf{i} - (\cos^2 t + \sin^2 t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t - \sin^2 t - \cos t \Rightarrow \int_C \mathbf{F} \cdot \mathbf{T} ds \\ &= \int_0^\pi (-\sin t \cos t - \sin^2 t - \cos t) dt = \left[-\frac{1}{2} \sin^2 t - \frac{t}{2} + \frac{\sin 2t}{4} - \sin t \right]_0^\pi = -\frac{\pi}{2}\end{aligned}$$

(b) $\mathbf{r} = (1-2t)\mathbf{i}$, $0 \leq t \leq 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i}$ and $\mathbf{F} = (1-2t)\mathbf{i} - (1-2t)^2\mathbf{j} \Rightarrow$

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t-1 \Rightarrow \int_C \mathbf{F} \cdot \mathbf{T} ds = \int_0^1 (2t-1) dt = [t^2 - t]_0^1 = 0$$

(c) $\mathbf{r}_1 = (1-t)\mathbf{i} - t\mathbf{j}$, $0 \leq t \leq 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_1}{dt} = -\mathbf{i} - \mathbf{j}$ and $\mathbf{F} = (1-2t)\mathbf{i} - (1-2t+2t^2)\mathbf{j}$

$$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}_1}{dt} = (2t-1) + (1-2t+2t^2) = 2t^2 \Rightarrow \text{Flow}_1 = \int_{C_1} \mathbf{F} \cdot \frac{d\mathbf{r}_1}{dt} = \int_0^1 2t^2 dt = \frac{2}{3}; \mathbf{r}_2 = -t\mathbf{i} + (t-1)\mathbf{j},$$

$$0 \leq t \leq 1, \text{ and } \mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_2}{dt} = -\mathbf{i} + \mathbf{j} \text{ and } \mathbf{F} = -\mathbf{i} - (t^2+t^2-2t+1)\mathbf{j}$$

$$= -\mathbf{i} - (2t^2-2t+1)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}_2}{dt} = 1 - (2t^2-2t+1) = 2t-2t^2 \Rightarrow \text{Flow}_2 = \int_{C_2} \mathbf{F} \cdot \frac{d\mathbf{r}_2}{dt} = \int_0^1 (2t-2t^2) dt$$

$$= \left[t^2 - \frac{2}{3}t^3 \right]_0^1 = \frac{1}{3} \Rightarrow \text{Flow} = \text{Flow}_1 + \text{Flow}_2 = \frac{2}{3} + \frac{1}{3} = 1$$

30. From $(1, 0)$ to $(0, 1)$: $\mathbf{r}_1 = (1-t)\mathbf{i} + t\mathbf{j}$, $0 \leq t \leq 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_1}{dt} = -\mathbf{i} + \mathbf{j}$,

$$\mathbf{F} = \mathbf{i} - (1-2t+2t^2)\mathbf{j}, \text{ and } \mathbf{n}_1 \cdot \mathbf{v}_1 = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_1 \cdot \mathbf{v}_1 = 2t-2t^2 \Rightarrow \text{Flux}_1 = \int_0^1 (2t-2t^2) dt \\ = \left[t^2 - \frac{2}{3}t^3 \right]_0^1 = \frac{1}{3};$$

From $(0, 1)$ to $(-1, 0)$: $\mathbf{r}_2 = -t\mathbf{i} + (1-t)\mathbf{j}$, $0 \leq t \leq 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_2}{dt} = -\mathbf{i} - \mathbf{j}$,

$$\mathbf{F} = (1-2t)\mathbf{i} - (1-2t+2t^2)\mathbf{j}, \text{ and } \mathbf{n}_2 \cdot \mathbf{v}_2 = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_2 \cdot \mathbf{v}_2 = (2t-1) + (-1+2t-2t^2) = -2+4t-2t^2$$

$$\Rightarrow \text{Flux}_2 = \int_0^1 (-2+4t-2t^2) dt = \left[-2t+2t^2 - \frac{2}{3}t^3 \right]_0^1 = -\frac{2}{3};$$

From $(-1, 0)$ to $(1, 0)$: $\mathbf{r}_3 = (-1+2t)\mathbf{i}$, $0 \leq t \leq 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_3}{dt} = 2\mathbf{i}$,

$$\mathbf{F} = (-1+2t)\mathbf{i} - (1-4t+4t^2)\mathbf{j}, \text{ and } \mathbf{n}_3 \cdot \mathbf{v}_3 = -2\mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_3 \cdot \mathbf{v}_3 = 2(1-4t+4t^2)$$

$$\Rightarrow \text{Flux}_3 = 2 \int_0^1 (1-4t+4t^2) dt = 2 \left[t - 2t^2 + \frac{4}{3}t^3 \right]_0^1 = \frac{2}{3} \Rightarrow \text{Flux} = \text{Flux}_1 + \text{Flux}_2 + \text{Flux}_3 = \frac{1}{3} - \frac{2}{3} + \frac{2}{3} = \frac{1}{3}$$

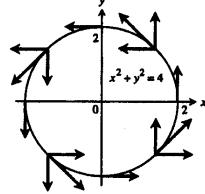
31. $\mathbf{F} = -\frac{y}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\mathbf{j}$ on $x^2+y^2=4$;

at $(2,0)$, $\mathbf{F} = \mathbf{j}$; at $(0,2)$, $\mathbf{F} = -\mathbf{i}$; at $(-2,0)$,

$$\mathbf{F} = -\mathbf{j}; \text{ at } (0,-2), \mathbf{F} = \mathbf{i}; \text{ at } \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \mathbf{F} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j};$$

$$\text{at } \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \mathbf{F} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}; \text{ at } \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right),$$

$$\mathbf{F} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}; \text{ at } \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \mathbf{F} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$



32. $\mathbf{F} = xi + yj$ on $x^2 + y^2 = 1$; at $(1,0)$, $\mathbf{F} = \mathbf{i}$;

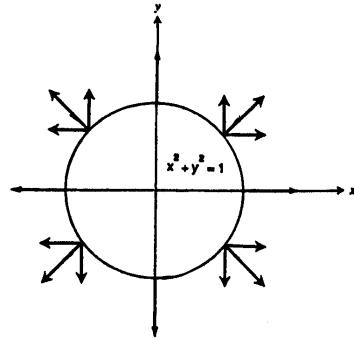
at $(-1,0)$, $\mathbf{F} = -\mathbf{i}$; at $(0,1)$, $\mathbf{F} = \mathbf{j}$; at $(0,-1)$,

$$\mathbf{F} = -\mathbf{j}; \text{ at } \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \mathbf{F} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j};$$

$$\text{at } \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \mathbf{F} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j};$$

$$\text{at } \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \mathbf{F} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}; \text{ at } \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right),$$

$$\mathbf{F} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}.$$



33. (a) $\mathbf{G} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is to have a magnitude $\sqrt{a^2+b^2}$ and to be tangent to $x^2+y^2=a^2+b^2$ in a counterclockwise direction. Thus $x^2+y^2=a^2+b^2 \Rightarrow 2x+2yy'=0 \Rightarrow y'=-\frac{x}{y}$ is the slope of the tangent line at any point on the circle $\Rightarrow y'=-\frac{a}{b}$ at (a,b) . Let $\mathbf{v}=-bi+aj \Rightarrow |\mathbf{v}|=\sqrt{a^2+b^2}$, with \mathbf{v} in a counterclockwise direction and tangent to the circle. Then let $P(x,y)=-y$ and $Q(x,y)=x$ $\Rightarrow \mathbf{G}=-yi+xj$ for (a,b) on $x^2+y^2=a^2+b^2$ we have $\mathbf{G}=-bi+aj$ and $|\mathbf{G}|=\sqrt{a^2+b^2}$.

(b) $\mathbf{G}=(\sqrt{x^2+y^2})\mathbf{F}=(\sqrt{a^2+b^2})\mathbf{F}$, since $x^2+y^2=a^2+b^2$

34. (a) From Exercise 33, part a, $-yi+xj$ is a vector tangent to the circle and pointing in a counterclockwise direction $\Rightarrow yi-xj$ is a vector tangent to the circle pointing in a clockwise direction $\Rightarrow \mathbf{G}=\frac{yi-xj}{\sqrt{x^2+y^2}}$ is a unit vector tangent to the circle and pointing in a clockwise direction.

(b) $\mathbf{G}=-\mathbf{F}$

35. The slope of the line through (x,y) and the origin is $\frac{y}{x} \Rightarrow \mathbf{v}=xi+yj$ is a vector parallel to that line and pointing away from the origin $\Rightarrow \mathbf{F}=-\frac{xi+yj}{\sqrt{x^2+y^2}}$ is the unit vector pointing toward the origin.

36. (a) From Exercise 35, $-\frac{\mathbf{x}\mathbf{i} + \mathbf{y}\mathbf{j}}{\sqrt{x^2 + y^2}}$ is a unit vector through (x, y) pointing toward the origin and we want

$$|\mathbf{F}| \text{ to have magnitude } \sqrt{x^2 + y^2} \Rightarrow \mathbf{F} = \sqrt{x^2 + y^2} \left(-\frac{\mathbf{x}\mathbf{i} + \mathbf{y}\mathbf{j}}{\sqrt{x^2 + y^2}} \right) = -x\mathbf{i} - y\mathbf{j}.$$

(b) We want $|\mathbf{F}| = \frac{C}{\sqrt{x^2 + y^2}} \Rightarrow \mathbf{F} = \frac{C}{\sqrt{x^2 + y^2}} \left(-\frac{\mathbf{x}\mathbf{i} + \mathbf{y}\mathbf{j}}{\sqrt{x^2 + y^2}} \right) = -C \left(\frac{\mathbf{x}\mathbf{i} + \mathbf{y}\mathbf{j}}{x^2 + y^2} \right)$, $C \neq 0$, and constant

$$37. \mathbf{F} = -4t^3\mathbf{i} + 8t^2\mathbf{j} + 2\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 12t^3 \Rightarrow \text{Flow} = \int_0^2 12t^3 dt = [3t^4]_0^2 = 48$$

$$38. \mathbf{F} = 12t^2\mathbf{j} + 9t^2\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = 3\mathbf{j} + 4\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 72t^2 \Rightarrow \text{Flow} = \int_0^1 72t^2 dt = [24t^3]_0^1 = 24$$

$$39. \mathbf{F} = (\cos t - \sin t)\mathbf{i} + (\cos t)\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t + 1$$

$$\Rightarrow \text{Flow} = \int_0^\pi (-\sin t \cos t + 1) dt = \left[\frac{1}{2} \cos^2 t + t \right]_0^\pi = \left(\frac{1}{2} + \pi \right) - \left(\frac{1}{2} + 0 \right) = \pi$$

$$40. \mathbf{F} = (-2 \sin t)\mathbf{i} - (2 \cos t)\mathbf{j} + 2\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = (2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -4 \sin^2 t - 4 \cos^2 t + 4 = 0$$

$$\Rightarrow \text{Flow} = 0$$

$$41. C_1: \mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, 0 \leq t \leq \frac{\pi}{2} \Rightarrow \mathbf{F} = (2 \cos t)\mathbf{i} + 2t\mathbf{j} + (2 \sin t)\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}$$

$$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -2 \cos t \sin t + 2t \cos t + 2 \sin t = -\sin 2t + 2t \cos t + 2 \sin t$$

$$\Rightarrow \text{Flow}_1 = \int_0^{\pi/2} (-\sin 2t + 2t \cos t + 2 \sin t) dt = \left[\frac{1}{2} \cos 2t + 2t \sin t + 2 \cos t - 2 \sin t \right]_0^{\pi/2} = -1 + \pi;$$

$$C_2: \mathbf{r} = \mathbf{j} + \frac{\pi}{2}(1-t)\mathbf{k}, 0 \leq t \leq 1 \Rightarrow \mathbf{F} = \pi(1-t)\mathbf{j} + 2\mathbf{k} \text{ and } \frac{d\mathbf{r}}{dt} = -\frac{\pi}{2}\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\pi$$

$$\Rightarrow \text{Flow}_2 = \int_0^1 -\pi dt = [-\pi t]_0^1 = -\pi;$$

$$C_3: \mathbf{r} = t\mathbf{i} + (1-t)\mathbf{j}, 0 \leq t \leq 1 \Rightarrow \mathbf{F} = 2t\mathbf{i} + 2(1-t)\mathbf{j} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t$$

$$\Rightarrow \text{Flow}_3 = \int_0^1 2t dt = [t^2]_0^1 = 1 \Rightarrow \text{Circulation} = (-1 + \pi) - \pi + 1 = 0$$

```

a:=0; b:= Pi;
M:= (x,y,z) -> 3/(1 + x^2);
N:= (x,y,z) -> 2/(1 + y^2);
P:= (x,y,z) -> 0;
F:= t -> vector([M(x(t),y(t),z(t)), N(x(t),y(t),z(t)), P(x(t),y(t),z(t))]);
dr:= t -> vector([D(x)(t), D(y)(t), D(z)(t)]);
integrand:= dotprod(F(t), dr(t), orthogonal);
int(integrand, t=a..b);
evalf(%);

```

Mathematica:

```

Clear[x,y,z,t]
r[t_] = {x[t],y[t],z[t]}
f[x_,y_] = {
  3/(1+x^2),
  2/(1+y^2)}
x[t_] = Cos[t]
y[t_] = Sin[t]
z[t_] = 0
{a,b} = {0,Pi};
v[t_] = r'[t]
integrand = f[x[t],y[t],z[t]] . v[t]
integrand = Simplify[integrand]

```

13.3 PATH INDEPENDENCE, POTENTIAL FUNCTIONS, AND CONSERVATIVE FIELDS

1. $\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = z = \frac{\partial M}{\partial y} \Rightarrow$ Conservative
2. $\frac{\partial P}{\partial y} = x \cos z = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = y \cos z = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = \sin z = \frac{\partial M}{\partial y} \Rightarrow$ Conservative
3. $\frac{\partial P}{\partial y} = -1 \neq 1 = \frac{\partial N}{\partial z} \Rightarrow$ Not Conservative
4. $\frac{\partial N}{\partial x} = 1 \neq -1 = \frac{\partial M}{\partial y} \Rightarrow$ Not Conservative
5. $\frac{\partial N}{\partial x} = 0 \neq 1 = \frac{\partial M}{\partial y} \Rightarrow$ Not Conservative
6. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = -e^x \sin y = \frac{\partial M}{\partial y} \Rightarrow$ Conservative
7. $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x,y,z) = x^2 + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 3y \Rightarrow g(y,z) = \frac{3y^2}{2} + h(z) \Rightarrow f(x,y,z) = x^2 + \frac{3y^2}{2} + h(z)$
 $\Rightarrow \frac{\partial f}{\partial z} = h'(z) = 4z \Rightarrow h(z) = 2z^2 + C \Rightarrow f(x,y,z) = x^2 + \frac{3y^2}{2} + 2z^2 + C$
8. $\frac{\partial f}{\partial x} = y + z \Rightarrow f(x,y,z) = (y+z)x + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = x + z \Rightarrow \frac{\partial g}{\partial y} = z \Rightarrow g(y,z) = zy + h(z)$
 $\Rightarrow f(x,y,z) = (y+z)x + zy + h(z) \Rightarrow \frac{\partial f}{\partial z} = x + y + h'(z) = x + y \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = (y+z)x + zy + C$
9. $\frac{\partial f}{\partial x} = e^{y+2z} \Rightarrow f(x,y,z) = xe^{y+2z} + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = xe^{y+2z} + \frac{\partial g}{\partial y} = xe^{y+2z} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow f(x,y,z) = xe^{y+2z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = 2xe^{y+2z} + h'(z) = 2xe^{y+2z} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = xe^{y+2z} + C$

$$10. \frac{\partial f}{\partial x} = y \sin z \Rightarrow f(x, y, z) = xy \sin z + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = x \sin z + \frac{\partial g}{\partial y} = x \sin z \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z)$$

$$\Rightarrow f(x, y, z) = xy \sin z + h(z) \Rightarrow \frac{\partial f}{\partial z} = xy \cos z + h'(z) = xy \cos z \Rightarrow h'(z) = 0 \Rightarrow h(z) + C \Rightarrow f(x, y, z) = xy \sin z + C$$

$$11. \frac{\partial f}{\partial z} = \frac{z}{y^2 + z^2} \Rightarrow f(x, y, z) = \frac{1}{2} \ln(y^2 + z^2) + g(x, y) \Rightarrow \frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} = \ln x + \sec^2(x + y) \Rightarrow g(x, y)$$

$$= (x \ln x - x) + \tan(x + y) + h(y) \Rightarrow f(x, y, z) = \frac{1}{2} \ln(y^2 + z^2) + (x \ln x - x) + \tan(x + y) + h(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{y}{y^2 + z^2} + \sec^2(x + y) + h'(y) = \sec^2(x + y) + \frac{y}{y^2 + z^2} \Rightarrow h'(y) = 0 \Rightarrow h(y) = C \Rightarrow f(x, y, z)$$

$$= \frac{1}{2} \ln(y^2 + z^2) + (x \ln x - x) + \tan(x + y) + C$$

$$12. \frac{\partial f}{\partial x} = \frac{y}{1+x^2y^2} \Rightarrow f(x, y, z) = \tan^{-1}(xy) + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{x}{1+x^2y^2} + \frac{\partial g}{\partial y} = \frac{x}{1+x^2y^2} + \frac{z}{\sqrt{1-y^2z^2}}$$

$$\Rightarrow \frac{\partial g}{\partial y} = \frac{z}{\sqrt{1-y^2z^2}} \Rightarrow g(y, z) = \sin^{-1}(yz) + h(z) \Rightarrow f(x, y, z) = \tan^{-1}(xy) + \sin^{-1}(yz) + h(z)$$

$$\Rightarrow \frac{\partial f}{\partial z} = \frac{y}{\sqrt{1-y^2z^2}} + h'(z) = \frac{y}{\sqrt{1-y^2z^2}} + \frac{1}{z} \Rightarrow h'(z) = \frac{1}{z} \Rightarrow h(z) = \ln|z| + C$$

$$\Rightarrow f(x, y, z) = \tan^{-1}(xy) + \sin^{-1}(yz) + \ln|z| + C$$

$$13. \text{ Let } \mathbf{F}(x, y, z) = 2xi + 2yj + 2zk \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow M dx + N dy + P dz \text{ is exact;}$$

$$\frac{\partial f}{\partial x} = 2x \Rightarrow f(x, y, z) = x^2 + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 2y \Rightarrow g(y, z) = y^2 + h(z) \Rightarrow f(x, y, z) = x^2 + y^2 + h(z)$$

$$\Rightarrow \frac{\partial f}{\partial z} = h'(z) = 2z \Rightarrow h(z) = z^2 + C \Rightarrow f(x, y, z) = x^2 + y^2 + z^2 + C \Rightarrow \int_{(0,0,0)}^{(2,3,-6)} 2x \, dx + 2y \, dy + 2z \, dz$$

$$= f(2, 3, -6) - f(0, 0, 0) = 2^2 + 3^2 + (-6)^2 = 49$$

$$14. \text{ Let } \mathbf{F}(x, y, z) = yzi + xzj + xyk \Rightarrow \frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = z = \frac{\partial M}{\partial y} \Rightarrow M dx + N dy + P dz \text{ is exact;}$$

$$\frac{\partial f}{\partial x} = yz \Rightarrow f(x, y, z) = xyz + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = xz + \frac{\partial g}{\partial y} = xz \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z)$$

$$= xyz + h(z) \Rightarrow \frac{\partial f}{\partial z} = xy + h'(z) = xy \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = xyz + C$$

$$\Rightarrow \int_{(1,1,2)}^{(3,5,0)} yz \, dx + xz \, dy + xy \, dz = f(3, 5, 0) - f(1, 1, 2) = 0 - 2 = -2$$

$$15. \text{ Let } \mathbf{F}(x, y, z) = 2xyi + (x^2 - z^2)j - 2yzk \Rightarrow \frac{\partial P}{\partial y} = -2z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 2x = \frac{\partial M}{\partial y}$$

$$\Rightarrow M dx + N dy + P dz \text{ is exact; } \frac{\partial f}{\partial x} = 2xy \Rightarrow f(x, y, z) = x^2y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 - z^2 \Rightarrow \frac{\partial g}{\partial y} = -z^2$$

$$\Rightarrow g(y, z) = -yz^2 + h(z) \Rightarrow f(x, y, z) = x^2y - yz^2 + h(z) \Rightarrow \frac{\partial f}{\partial z} = -2yz + h'(z) = -2yz \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$$

$$\Rightarrow f(x, y, z) = x^2y - yz^2 + C \Rightarrow \int_{(0,0,0)}^{(1,2,3)} 2xy \, dx + (x^2 - z^2) \, dy - 2yz \, dz = f(1, 2, 3) - f(0, 0, 0) = 2 - 2(3)^2 = -16$$

16. Let $\mathbf{F}(x, y, z) = 2xi - y^2j - \left(\frac{4}{1+z^2}\right)k \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y}$

$\Rightarrow M \, dx + N \, dy + P \, dz$ is exact; $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x, y, z) = x^2 + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = -y^2 \Rightarrow g(y, z) = -\frac{y^3}{3} + h(z)$

$\Rightarrow f(x, y, z) = x^2 - \frac{y^3}{3} + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = -\frac{4}{1+z^2} \Rightarrow h(z) = -4 \tan^{-1} z + C \Rightarrow f(x, y, z)$

$= x^2 - \frac{y^3}{3} - 4 \tan^{-1} z + C \Rightarrow \int_{(0,0,0)}^{(3,3,1)} 2x \, dx - y^2 \, dy - \frac{4}{1+z^2} \, dz = f(3, 3, 1) - f(0, 0, 0)$

$= \left(9 - \frac{27}{3} - 4 \cdot \frac{\pi}{4} + C\right) - (0 - 0 - 0 + C) = -\pi$

17. Let $\mathbf{F}(x, y, z) = (\sin y \cos x)i + (\cos y \sin x)j + k \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \cos y \cos x = \frac{\partial M}{\partial y}$

$\Rightarrow M \, dx + N \, dy + P \, dz$ is exact; $\frac{\partial f}{\partial x} = \sin y \cos x \Rightarrow f(x, y, z) = \sin y \sin x + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \cos y \sin x + \frac{\partial g}{\partial y}$

$= \cos y \sin x \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z) = \sin y \sin x + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 1 \Rightarrow h(z) = z + C$

$\Rightarrow f(x, y, z) = \sin y \sin x + z + C \Rightarrow \int_{(1,0,0)}^{(0,1,1)} \sin y \cos x \, dx + \cos y \sin x \, dy + dz = f(0, 1, 1) - f(1, 0, 0)$

$= (0 + 1 + C) - (0 + 0 + C) = 1$

18. Let $\mathbf{F}(x, y, z) = (2 \cos y)i + \left(\frac{1}{y} - 2x \sin y\right)j + \left(\frac{1}{z}\right)k \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -2 \sin y = \frac{\partial M}{\partial y}$

$\Rightarrow M \, dx + N \, dy + P \, dz$ is exact; $\frac{\partial f}{\partial x} = 2 \cos y \Rightarrow f(x, y, z) = 2x \cos y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = -2x \sin y + \frac{\partial g}{\partial y}$

$= \frac{1}{y} - 2x \sin y \Rightarrow \frac{\partial g}{\partial y} = \frac{1}{y} \Rightarrow g(y, z) = \ln |y| + h(z) \Rightarrow f(x, y, z) = 2x \cos y + \ln |y| + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = \frac{1}{z}$

$\Rightarrow h(z) = \ln |z| + C \Rightarrow f(x, y, z) = 2x \cos y + \ln |y| + \ln |z| + C$

$\Rightarrow \int_{(0,2,1)}^{(1,\pi/2,2)} 2 \cos y \, dx + \left(\frac{1}{y} - 2x \sin y\right) \, dy + \frac{1}{z} \, dz = f\left(1, \frac{\pi}{2}, 2\right) - f(0, 2, 1)$

$= \left(2 \cdot 0 + \ln \frac{\pi}{2} + \ln 2 + C\right) - (0 \cdot \cos 2 + \ln 2 + \ln 1 + C) = \ln \frac{\pi}{2}$

19. Let $\mathbf{F}(x, y, z) = 3x^2i + \left(\frac{z^2}{y}\right)j + (2z \ln y)k \Rightarrow \frac{\partial P}{\partial y} = \frac{2z}{y} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y}$

$\Rightarrow M \, dx + N \, dy + P \, dz$ is exact; $\frac{\partial f}{\partial x} = 3x^2 \Rightarrow f(x, y, z) = x^3 + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = \frac{z^2}{y} \Rightarrow g(y, z) = z^2 \ln y + h(z)$

$\Rightarrow f(x, y, z) = x^3 + z^2 \ln y + h(z) \Rightarrow \frac{\partial f}{\partial z} = 2z \ln y + h'(z) = 2z \ln y \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z)$

$$= x^3 + z^2 \ln y + C \Rightarrow \int_{(1,1,1)}^{(1,2,3)} 3x^2 dx + \frac{z^2}{y} dy + 2z \ln y dz = f(1,2,3) - f(1,1,1)$$

$$= (1 + 9 \ln 2 + C) - (1 + 0 + C) = 9 \ln 2$$

20. Let $\mathbf{F}(x,y,z) = (2x \ln y - yz)\mathbf{i} + \left(\frac{x^2}{y} - xz\right)\mathbf{j} - (xy)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -x = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{2x}{y} - z = \frac{\partial M}{\partial y}$
 $\Rightarrow M dx + N dy + P dz$ is exact; $\frac{\partial f}{\partial x} = 2x \ln y - yz \Rightarrow f(x,y,z) = x^2 \ln y - xyz + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{x^2}{y} - xz + \frac{\partial g}{\partial y}$
 $= \frac{x^2}{y} - xz \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z) \Rightarrow f(x,y,z) = x^2 \ln y - xyz + h(z) \Rightarrow \frac{\partial f}{\partial z} = -xy + h'(z) = -xy \Rightarrow h'(z) = 0$
 $\Rightarrow h(z) = C \Rightarrow f(x,y,z) = x^2 \ln y - xyz + C \Rightarrow \int_{(1,2,1)}^{(2,1,1)} (2x \ln y - yz) dx + \left(\frac{x^2}{y} - xz\right) dy - xy dz$

$$= f(2,1,1) - f(1,2,1) = (4 \ln 1 - 2 + C) - (\ln 2 - 2 + C) = -\ln 2$$

21. Let $\mathbf{F}(x,y,z) = \left(\frac{1}{y}\right)\mathbf{i} + \left(\frac{1}{z} - \frac{x}{y^2}\right)\mathbf{j} - \left(\frac{y}{z^2}\right)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -\frac{1}{z^2} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{1}{y^2} = \frac{\partial M}{\partial y}$
 $\Rightarrow M dx + N dy + P dz$ is exact; $\frac{\partial f}{\partial x} = \frac{1}{y} \Rightarrow f(x,y,z) = \frac{x}{y} + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = -\frac{x}{y^2} + \frac{\partial g}{\partial y} = \frac{1}{z} - \frac{x}{y^2}$
 $\Rightarrow \frac{\partial g}{\partial y} = \frac{1}{z} \Rightarrow g(y,z) = \frac{y}{z} + h(z) \Rightarrow f(x,y,z) = \frac{x}{y} + \frac{y}{z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = -\frac{y}{z^2} + h'(z) = -\frac{y}{z^2} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$
 $\Rightarrow f(x,y,z) = \frac{x}{y} + \frac{y}{z} + C \Rightarrow \int_{(1,1,1)}^{(2,2,2)} \frac{1}{y} dx + \left(\frac{1}{z} - \frac{x}{y^2}\right) dy - \frac{y}{z^2} dz = f(2,2,2) - f(1,1,1) = \left(\frac{2}{2} + \frac{2}{2} + C\right) - \left(\frac{1}{1} + \frac{1}{1} + C\right)$
 $= 0$

22. Let $\mathbf{F}(x,y,z) = \frac{2xi + 2yj + 2zk}{x^2 + y^2 + z^2}$ (and let $\rho^2 = x^2 + y^2 + z^2 \Rightarrow \frac{\partial \rho}{\partial x} = \frac{x}{\rho}, \frac{\partial \rho}{\partial y} = \frac{y}{\rho}, \frac{\partial \rho}{\partial z} = \frac{z}{\rho}$)
 $\Rightarrow \frac{\partial P}{\partial y} = -\frac{4yz}{\rho^4} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{4xz}{\rho^4} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{4xy}{\rho^4} = \frac{\partial M}{\partial y} \Rightarrow M dx + N dy + P dz$ is exact;
 $\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2 + z^2} \Rightarrow f(x,y,z) = \ln(x^2 + y^2 + z^2) + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2 + z^2} + \frac{\partial g}{\partial y} = \frac{2y}{x^2 + y^2 + z^2}$
 $\Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z) \Rightarrow f(x,y,z) = \ln(x^2 + y^2 + z^2) + h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{2z}{x^2 + y^2 + z^2} + h'(z)$
 $= \frac{2z}{x^2 + y^2 + z^2} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = \ln(x^2 + y^2 + z^2) + C$
 $\Rightarrow \int_{(-1,-1,-1)}^{(2,2,2)} \frac{2x dx + 2y dy + 2z dz}{x^2 + y^2 + z^2} = f(2,2,2) - f(-1,-1,-1) = \ln 12 - \ln 3 = \ln 4$

23. $\mathbf{r} = (i + j + k) + t(i + 2j - 2k) = (1 + t)i + (1 + 2t)j + (1 - 2t)k \Rightarrow dx = dt, dy = 2 dt, dz = -2 dt$
 $\Rightarrow \int_{(1,1,1)}^{(2,3,-1)} y dx + x dy + 4 dz = \int_0^1 (2t + 1) dt + (t + 1)(2 dt) + 4(-2) dt = \int_0^1 (4t - 5) dt = [2t^2 - 5t]_0^1 = -3$

24. $\mathbf{r} = t(3\mathbf{j} + 4\mathbf{k}) \Rightarrow dx = 0, dy = 3 dt, dz = 4 dt \Rightarrow \int_{(0,0,0)}^{(0,3,4)} x^2 dx + yz dy + \left(\frac{y^2}{2}\right) dz$
 $= \int_0^1 (12t^2)(3 dt) + \left(\frac{9t^2}{2}\right)(4 dt) = \int_0^1 54t^2 dt = [18t^2]_0^1 = 18$

25. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow M dx + N dy + P dz \text{ is exact} \Rightarrow \mathbf{F} \text{ is conservative}$
 $\Rightarrow \text{path independence}$

26. $\frac{\partial P}{\partial y} = -\frac{yz}{(\sqrt{x^2+y^2+z^2})^3} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{xz}{(\sqrt{x^2+y^2+z^2})^3} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{xy}{(\sqrt{x^2+y^2+z^2})^3} = \frac{\partial M}{\partial y}$
 $\Rightarrow M dx + N dy + P dz \text{ is exact} \Rightarrow \mathbf{F} \text{ is conservative} \Rightarrow \text{path independence}$

27. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{2x}{y^2} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} \text{ is conservative} \Rightarrow \text{there exists an } f \text{ so that } \mathbf{F} = \nabla f;$
 $\frac{\partial f}{\partial x} = \frac{2x}{y} \Rightarrow f(x,y) = \frac{x^2}{y} + g(y) \Rightarrow \frac{\partial f}{\partial y} = -\frac{x^2}{y^2} + g'(y) = \frac{1-x^2}{y^2} \Rightarrow g'(y) = \frac{1}{y^2} \Rightarrow g(y) = -\frac{1}{y} + C$
 $\Rightarrow f(x,y) = \frac{x^2}{y} - \frac{1}{y} + C \Rightarrow \mathbf{F} = \nabla \left(\frac{x^2-1}{y} \right)$

28. $\frac{\partial P}{\partial y} = \cos z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{e^x}{y} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} \text{ is conservative} \Rightarrow \text{there exists an } f \text{ so that } \mathbf{F} = \nabla f;$
 $\frac{\partial f}{\partial x} = e^x \ln y \Rightarrow f(x,y,z) = e^x \ln y + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{e^x}{y} + \frac{\partial g}{\partial y} = \frac{e^x}{y} + \sin z \Rightarrow \frac{\partial g}{\partial y} = \sin z \Rightarrow g(y,z) =$
 $= y \sin z + h(z) \Rightarrow f(x,y,z) = e^x \ln y + y \sin z + h(z) \Rightarrow \frac{\partial f}{\partial z} = y \cos z + h'(z) = y \cos z \Rightarrow h'(z) = 0$
 $\Rightarrow h(z) = C \Rightarrow f(x,y,z) = e^x \ln y + y \sin z + C \Rightarrow \mathbf{F} = \nabla(e^x \ln y + y \sin z)$

29. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} \text{ is conservative} \Rightarrow \text{there exists an } f \text{ so that } \mathbf{F} = \nabla f;$
 $\frac{\partial f}{\partial x} = x^2 + y \Rightarrow f(x,y,z) = \frac{1}{3}x^3 + xy + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = y^2 + x \Rightarrow \frac{\partial g}{\partial y} = y^2 \Rightarrow g(y,z) = \frac{1}{3}y^3 + h(z)$
 $\Rightarrow f(x,y,z) = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = ze^z \Rightarrow h(z) = ze^z - e^z + C \Rightarrow f(x,y,z)$
 $= \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z + C \Rightarrow \mathbf{F} = \nabla \left(\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z \right)$

(a) work = $\int_A^B \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z \right]_{(1,0,0)}^{(1,0,1)} = \left(\frac{1}{3} + 0 + 0 + e - e \right) - \left(\frac{1}{3} + 0 + 0 - 1 \right)$
 $= 1$

(b) work = $\int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z \right]_{(1,0,0)}^{(1,0,1)} = 1$

(c) work = $\int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z \right]_{(1,0,0)}^{(1,0,1)} = 1$

Note: Since \mathbf{F} is conservative, $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path from $(1, 0, 0)$ to $(1, 0, 1)$.

30. $\frac{\partial P}{\partial y} = xe^{yz} + xyze^{yz} + \cos y = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = ye^{yz} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = ze^{yz} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative \Rightarrow there exists an f so that $\mathbf{F} = \nabla f; \frac{\partial f}{\partial x} = e^{yz} \Rightarrow f(x, y, z) = xe^{yz} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = xze^{yz} + \frac{\partial g}{\partial y} = xze^{yz} + z \cos y \Rightarrow \frac{\partial g}{\partial y} = z \cos y$
 $\Rightarrow g(y, z) = z \sin y + h(z) \Rightarrow f(x, y, z) = xe^{yz} + z \sin y + h(z) \Rightarrow \frac{\partial f}{\partial z} = xye^{yz} + \sin y + h'(z) = xye^{yz} + \sin y$
 $\Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = xe^{yz} + z \sin y + C \Rightarrow \mathbf{F} = \nabla(xe^{yz} + z \sin y)$

$$(a) \text{ work} = \int_A^B \mathbf{F} \cdot d\mathbf{r} = [xe^{yz} + z \sin y]_{(1, 0, 1)}^{(1, \pi/2, 0)} = (1 + 0) - (1 + 0) = 0$$

$$(b) \text{ work} = \int_A^B \mathbf{F} \cdot d\mathbf{r} = [xe^{yz} + z \sin y]_{(1, 0, 1)}^{(1, \pi/2, 0)} = 0$$

$$(c) \text{ work} = \int_A^B \mathbf{F} \cdot d\mathbf{r} = [xe^{yz} + z \sin y]_{(1, 0, 1)}^{(1, \pi/2, 0)} = 0$$

Note: Since \mathbf{F} is conservative, $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path from $(1, 0, 1)$ to $(1, \frac{\pi}{2}, 0)$.

31. (a) $\mathbf{F} = \nabla(x^3y^2) \Rightarrow \mathbf{F} = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$; let C_1 be the path from $(-1, 1)$ to $(0, 0) \Rightarrow x = t - 1$ and $y = -t + 1, 0 \leq t \leq 1 \Rightarrow \mathbf{F} = 3(t-1)^2(-t+1)^2\mathbf{i} + 2(t-1)^3(-t+1)\mathbf{j} = 3(t-1)^4\mathbf{i} - 2(t-1)^4\mathbf{j}$

$$\text{and } \mathbf{r}_1 = (t-1)\mathbf{i} + (-t+1)\mathbf{j} \Rightarrow d\mathbf{r}_1 = dt\mathbf{i} - dt\mathbf{j} \Rightarrow \int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 = \int_0^1 [3(t-1)^4 + 2(t-1)^4] dt$$

$$= \int_0^1 5(t-1)^4 dt = [(t-1)^5]_0^1 = 1; \text{ let } C_2 \text{ be the path from } (0, 0) \text{ to } (1, 1) \Rightarrow x = t \text{ and } y = t,$$

$$0 \leq t \leq 1 \Rightarrow \mathbf{F} = 3t^4\mathbf{i} + 2t^4\mathbf{j} \text{ and } \mathbf{r}_2 = t\mathbf{i} + t\mathbf{j} \Rightarrow d\mathbf{r}_2 = dt\mathbf{i} + dt\mathbf{j} \Rightarrow \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 = \int_0^1 (3t^4 + 2t^4) dt$$

$$= \int_0^1 5t^4 dt = 1 \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 = 2$$

- (b) Since $f(x, y) = x^3y^2$ is a potential function for \mathbf{F} , $\int_{(-1, 1)}^{(1, 1)} \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(-1, 1) = 2$

32. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -2x \sin y = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative \Rightarrow there exists an f so that $\mathbf{F} = \nabla f$;

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2x \cos y \Rightarrow f(x, y, z) = x^2 \cos y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = -x^2 \sin y + \frac{\partial g}{\partial y} = -x^2 \sin y \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \\ \Rightarrow f(x, y, z) &= x^2 \cos y + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = x^2 \cos y + C \Rightarrow \mathbf{F} = \nabla(x^2 \cos y)\end{aligned}$$

(a) $\int_C 2x \cos y \, dx - x^2 \sin y \, dy = [x^2 \cos y]_{(1,0)}^{(0,1)} = 0 - 1 = -1$

(b) $\int_C 2x \cos y \, dx - x^2 \sin y \, dy = [x^2 \cos y]_{(-1,\pi)}^{(1,0)} = 1 - (-1) = 2$

(c) $\int_C 2x \cos y \, dx - x^2 \sin y \, dy = [x^2 \cos y]_{(-1,0)}^{(1,0)} = 1 - 1 = 0$

(d) $\int_C 2x \cos y \, dx - x^2 \sin y \, dy = [x^2 \cos y]_{(1,0)}^{(1,0)} = 1 - 1 = 0$

33. (a) If the differential form is exact, then $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \Rightarrow 2ay = cy$ for all $y \Rightarrow 2a = c$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \Rightarrow 2cx = 2cx$ for

all x , and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow by = 2ay$ for all $y \Rightarrow b = 2a$ and $c = 2a$

(b) $\mathbf{F} = \nabla f \Rightarrow$ the differential form with $a = 1$ in part (a) is exact $\Rightarrow b = 2$ and $c = 2$

34. $\mathbf{F} = \nabla f \Rightarrow g(x, y, z) = \int_{(0,0,0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{r} = \int_{(0,0,0)}^{(x,y,z)} \nabla f \cdot d\mathbf{r} = f(x, y, z) - f(0, 0, 0) \Rightarrow \frac{\partial g}{\partial x} = \frac{\partial f}{\partial x} - 0, \frac{\partial g}{\partial y} = \frac{\partial f}{\partial y} - 0$, and

$$\frac{\partial g}{\partial z} = \frac{\partial f}{\partial z} - 0 \Rightarrow \nabla g = \nabla f = \mathbf{F}, \text{ as claimed}$$

35. The path will not matter; the work along any path will be the same because the field is conservative.

36. The field is not conservative, for otherwise the work would be the same along C_1 and C_2 .

37. Let the coordinates of points A and B be (x_A, y_A, z_A) and (x_B, y_B, z_B) , respectively. The force $\mathbf{F} = ai + bj + ck$ is conservative because all the partial derivatives of M, N, and P are zero. Therefore, the potential function is $f(x, y, z) = ax + by + cz + C$, and the work done by the force in moving a particle along any path from A to B is $f(B) - f(A) = f(x_B, y_B, z_B) - f(x_A, y_A, z_A) = (ax_B + by_B + cz_B + C) - (ax_A + by_A + cz_A + C)$
 $= a(x_B - x_A) + b(y_B - y_A) + c(z_B - z_A) = \mathbf{F} \cdot \vec{AB}$

38. (a) Let $-GmM = C \Rightarrow \mathbf{F} = C \left[\frac{x}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{i} + \frac{y}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{j} + \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{k} \right]$

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{-3yzC}{(x^2 + y^2 + z^2)^{5/2}} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = \frac{-3xzC}{(x^2 + y^2 + z^2)^{5/2}} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{-3xyC}{(x^2 + y^2 + z^2)^{5/2}} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$$

$$= \nabla f \text{ for some } f; \frac{\partial f}{\partial x} = \frac{xC}{(x^2 + y^2 + z^2)^{3/2}} \Rightarrow f(x, y, z) = -\frac{C}{(x^2 + y^2 + z^2)^{1/2}} + g(y, z)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{yC}{(x^2 + y^2 + z^2)^{3/2}} + \frac{\partial g}{\partial y} = \frac{yC}{(x^2 + y^2 + z^2)^{3/2}} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z)$$

$$\Rightarrow \frac{\partial f}{\partial z} = \frac{zC}{(x^2 + y^2 + z^2)^{3/2}} + h'(z) = \frac{zC}{(x^2 + y^2 + z^2)^{3/2}} \Rightarrow h(z) = C_1 \Rightarrow f(x, y, z) = -\frac{C}{(x^2 + y^2 + z^2)^{1/2}} + C_1.$$

Let $C_1 = 0 \Rightarrow f(x, y, z) = \frac{GmM}{(x^2 + y^2 + z^2)^{1/2}}$ is a potential function for \mathbf{F} .

(b) If s is the distance of (x, y, z) from the origin, then $s = \sqrt{x^2 + y^2 + z^2}$. The work done by the gravitational

field \mathbf{F} is work $= \int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r} = \left[\frac{GmM}{\sqrt{x^2 + y^2 + z^2}} \right]_{P_1}^{P_2} = \frac{GmM}{s_2} - \frac{GmM}{s_1} = GmM \left(\frac{1}{s_2} - \frac{1}{s_1} \right)$, as claimed.

13.4 GREEN'S THEOREM IN THE PLANE

1. $M = -y = -a \sin t$, $N = x = a \cos t$, $dx = -a \sin t dt$, $dy = a \cos t dt \Rightarrow \frac{\partial M}{\partial x} = 0$, $\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = 1$, and $\frac{\partial N}{\partial y} = 0$;

Equation (3): $\oint_C M dy - N dx = \int_0^{2\pi} [(-a \sin t)(a \cos t) - (a \cos t)(-a \sin t)] dt = \int_0^{2\pi} 0 dt = 0$;

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy = \iint_R 0 dx dy = 0, \text{ Flux}$$

Equation (4): $\oint_C M dx + N dy = \int_0^{2\pi} [(-a \sin t)(-a \sin t) + (a \cos t)(a \cos t)] dt = \int_0^{2\pi} a^2 dt = 2\pi a^2$;

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \int_{-a}^a \int_{-\sqrt{a^2 - y^2}}^{\sqrt{a^2 - y^2}} 2 dx dy = \int_{-a}^a 4\sqrt{a^2 - y^2} dy = 4 \left[\frac{y}{2} \sqrt{a^2 - y^2} + \frac{a^2}{2} \sin^{-1} \frac{y}{a} \right]_{-a}^a$$

$$= 2a^2 \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = 2a^2 \pi, \text{ Circulation}$$

2. $M = y = a \sin t$, $N = 0$, $dx = -a \sin t dt$, $dy = a \cos t dt \Rightarrow \frac{\partial M}{\partial x} = 0$, $\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = 0$, and $\frac{\partial N}{\partial y} = 0$;

Equation (3): $\oint_C M dy - N dx = \int_0^{2\pi} a^2 \sin t \cos t dt = a^2 \left[\frac{1}{2} \sin^2 t \right]_0^{2\pi} = 0$; $\iint_R 0 dx dy = 0$, Flux

Equation (4): $\oint_C M dx + N dy = \int_0^{2\pi} (-a^2 \sin^2 t) dt = -a^2 \left[\frac{t}{2} - \frac{\sin 2t}{4} \right]_0^{2\pi} = -\pi a^2$; $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

$$= \int_R \int -1 \, dx \, dy = \int_0^{2\pi} \int_0^a -r \, dr \, d\theta = \int_0^{2\pi} -\frac{a^2}{2} \, d\theta = -\pi a^2, \text{ Circulation}$$

3. $M = 2x = 2a \cos t$, $N = -3y = -3a \sin t$, $dx = -a \sin t \, dt$, $dy = a \cos t \, dt \Rightarrow \frac{\partial M}{\partial x} = 2$, $\frac{\partial M}{\partial y} = 0$, $\frac{\partial N}{\partial x} = 0$, and $\frac{\partial N}{\partial y} = -3$;

$$\begin{aligned} \text{Equation (3): } & \oint_C M \, dy - N \, dx = \int_0^{2\pi} [(2a \cos t)(a \cos t) + (3a \sin t)(-a \sin t)] \, dt \\ &= \int_0^{2\pi} (2a^2 \cos^2 t - 3a^2 \sin^2 t) \, dt = 2a^2 \left[\frac{t}{2} + \frac{\sin 2t}{4} \right]_0^{2\pi} - 3a^2 \left[\frac{t}{2} - \frac{\sin 2t}{4} \right]_0^{2\pi} = 2\pi a^2 - 3\pi a^2 = -\pi a^2; \\ \int_R \int \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) &= \int_R \int -1 \, dx \, dy = \int_0^{2\pi} \int_0^a -r \, dr \, d\theta = \int_0^{2\pi} -\frac{a^2}{2} \, d\theta = -\pi a^2, \text{ Flux} \\ \text{Equation (4): } & \oint_C M \, dx + N \, dy = \int_0^{2\pi} [(2a \cos t)(-a \sin t) + (-3a \sin t)(a \cos t)] \, dt \\ &= \int_0^{2\pi} (-2a^2 \sin t \cos t - 3a^2 \sin t \cos t) \, dt = -5a^2 \left[\frac{1}{2} \sin^2 t \right]_0^{2\pi} = 0; \int_R \int 0 \, dx \, dy = 0, \text{ Circulation} \end{aligned}$$

4. $M = -x^2y = -a^3 \cos^2 t \sin t$, $N = xy^2 = a^3 \cos t \sin^2 t$, $dx = -a \sin t \, dt$, $dy = a \cos t \, dt$

$$\Rightarrow \frac{\partial M}{\partial x} = -2a^2 \cos t \sin t$$
, $\frac{\partial M}{\partial y} = -a^2 \cos^2 t$, $\frac{\partial N}{\partial x} = a^2 \sin^2 t$, and $\frac{\partial N}{\partial y} = 2a^2 \cos t \sin t$;

$$\begin{aligned} \text{Equation (3): } & \oint_C M \, dy - N \, dx = \int_0^{2\pi} (-a^4 \cos^3 t \sin t + a^4 \cos t \sin^3 t) = \left[\frac{a^4}{4} \cos^4 t + \frac{a^4}{4} \sin^4 t \right]_0^{2\pi} = 0; \\ \int_R \int \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) &= \int_R \int (-2xy + 2xy) \, dx \, dy = 0, \text{ Flux} \\ \text{Equation (4): } & \oint_C M \, dx + N \, dy = \int_0^{2\pi} (a^4 \cos^2 t \sin^2 t + a^4 \cos^2 t \sin^2 t) \, dt = \int_0^{2\pi} (2a^4 \cos^2 t \sin^2 t) \, dt \\ &= \int_0^{2\pi} \frac{1}{2} a^4 \sin^2 2t \, dt = \frac{a^4}{4} \int_0^{4\pi} \sin^2 u \, du = \frac{a^4}{4} \left[\frac{u}{2} - \frac{\sin 2u}{4} \right]_0^{4\pi} = \frac{\pi a^4}{2}; \int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) &= \int_R \int (y^2 + x^2) \, dx \, dy \\ &= \int_0^{2\pi} \int_0^a r^2 \cdot r \, dr \, d\theta = \int_0^{2\pi} \frac{a^4}{4} \, d\theta = \frac{\pi a^4}{2}, \text{ Circulation} \end{aligned}$$

5. $M = x - y$, $N = y - x \Rightarrow \frac{\partial M}{\partial x} = 1$, $\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = -1$, $\frac{\partial N}{\partial y} = 1 \Rightarrow$ Flux = $\int_R \int 2 \, dx \, dy = \int_0^1 \int_0^1 2 \, dx \, dy = 2$;

$$\text{Circ} = \int_R \int [-1 - (-1)] \, dx \, dy = 0$$

$$\begin{aligned}
6. \quad M &= x^2 + 4y, N = x + y^2 \Rightarrow \frac{\partial M}{\partial x} = 2x, \frac{\partial M}{\partial y} = 4, \frac{\partial N}{\partial x} = 1, \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R (2x + 2y) dx dy \\
&= \int_0^1 \int_0^1 (2x + 2y) dx dy = \int_0^1 [x^2 + 2xy]_0^1 dy = \int_0^1 (1 + 2y) dy = [y + y^2]_0^1 = 2; \text{ Circ} = \iint_R (1 - 4) dx dy \\
&= \int_0^1 \int_0^1 -3 dx dy = -3
\end{aligned}$$

$$\begin{aligned}
7. \quad M &= y^2 - x^2, N = x^2 + y^2 \Rightarrow \frac{\partial M}{\partial x} = -2x, \frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = 2x, \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R (-2x + 2y) dx dy \\
&= \int_0^3 \int_0^x (-2x + 2y) dy dx = \int_0^3 (-2x^2 + x^2) dx = \left[-\frac{1}{3}x^3 \right]_0^3 = -9; \text{ Circ} = \iint_R (2x - 2y) dx dy \\
&= \int_0^3 \int_0^x (2x - 2y) dy dx = \int_0^3 x^2 dx = 9
\end{aligned}$$

$$\begin{aligned}
8. \quad M &= x + y, N = -(x^2 + y^2) \Rightarrow \frac{\partial M}{\partial x} = 1, \frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x} = -2x, \frac{\partial N}{\partial y} = -2y \Rightarrow \text{Flux} = \iint_R (1 - 2y) dx dy \\
&= \int_0^1 \int_0^x (1 - 2y) dy dx = \int_0^1 (x - x^2) dx = \frac{1}{6}; \text{ Circ} = \iint_R (-2x - 1) dx dy = \int_0^1 \int_0^x (-2x - 1) dy dx \\
&= \int_0^1 (-2x^2 - x) dx = -\frac{7}{6}
\end{aligned}$$

$$\begin{aligned}
9. \quad M &= x + e^x \sin y, N = x + e^x \cos y \Rightarrow \frac{\partial M}{\partial x} = 1 + e^x \sin y, \frac{\partial M}{\partial y} = e^x \cos y, \frac{\partial N}{\partial x} = 1 + e^x \cos y, \frac{\partial N}{\partial y} = -e^x \sin y \\
\Rightarrow \text{Flux} &= \iint_R dx dy = \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r dr d\theta = \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2} \cos 2\theta \right) d\theta = \left[\frac{1}{4} \sin 2\theta \right]_{-\pi/4}^{\pi/4} = \frac{1}{2}; \\
\text{Circ} &= \iint_R (1 + e^x \cos y - e^x \cos y) dx dy = \iint_R dx dy = \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r dr d\theta = \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2} \cos 2\theta \right) d\theta = \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
10. \quad M &= \tan^{-1} \frac{y}{x}, N = \ln(x^2 + y^2) \Rightarrow \frac{\partial M}{\partial x} = \frac{-y}{x^2 + y^2}, \frac{\partial M}{\partial y} = \frac{x}{x^2 + y^2}, \frac{\partial N}{\partial x} = \frac{2x}{x^2 + y^2}, \frac{\partial N}{\partial y} = \frac{2y}{x^2 + y^2} \\
\Rightarrow \text{Flux} &= \iint_R \left(\frac{-y}{x^2 + y^2} + \frac{2y}{x^2 + y^2} \right) dx dy = \int_0^\pi \int_1^2 \left(\frac{r \sin \theta}{r^2} \right) r dr d\theta = \int_0^\pi \sin \theta d\theta = 2; \\
\text{Circ} &= \iint_R \left(\frac{2x}{x^2 + y^2} - \frac{x}{x^2 + y^2} \right) dx dy = \int_0^\pi \int_1^2 \left(\frac{r \cos \theta}{r^2} \right) r dr d\theta = \int_0^\pi \cos \theta d\theta = 0
\end{aligned}$$

$$11. M = xy, N = y^2 \Rightarrow \frac{\partial M}{\partial x} = y, \frac{\partial M}{\partial y} = x, \frac{\partial N}{\partial x} = 0, \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R (y + 2y) dy dx = \int_0^1 \int_{x^2}^x 3y dy dx$$

$$= \int_0^1 \left(\frac{3x^2}{2} - \frac{3x^4}{2} \right) dx = \frac{1}{5}; \text{ Circ} = \iint_R -x dy dx = \int_0^1 \int_{x^2}^x -x dy dx = \int_0^1 (-x^2 + x^3) dx = -\frac{1}{12}$$

$$12. M = -\sin y, N = x \cos y \Rightarrow \frac{\partial M}{\partial x} = 0, \frac{\partial M}{\partial y} = -\cos y, \frac{\partial N}{\partial x} = \cos y, \frac{\partial N}{\partial y} = -x \sin y$$

$$\Rightarrow \text{Flux} = \iint_R (-x \sin y) dx dy = \int_0^{\pi/2} \int_0^{\pi/2} (-x \sin y) dx dy = \int_0^{\pi/2} \left(-\frac{\pi^2}{8} \sin y \right) dy = -\frac{\pi^2}{8};$$

$$\text{Circ} = \iint_R [\cos y - (-\cos y)] dx dy = \int_0^{\pi/2} \int_0^{\pi/2} 2 \cos y dx dy = \int_0^{\pi/2} \pi \cos y dy = [\pi \sin y]_0^{\pi/2} = \pi$$

$$13. M = 3xy - \frac{x}{1+y^2}, N = e^x + \tan^{-1} y \Rightarrow \frac{\partial M}{\partial x} = 3y - \frac{1}{1+y^2}, \frac{\partial N}{\partial y} = \frac{1}{1+y^2}$$

$$\Rightarrow \text{Flux} = \iint_R \left(3y - \frac{1}{1+y^2} + \frac{1}{1+y^2} \right) dx dy = \iint_R 3y dx dy = \int_0^{2\pi} \int_0^{a(1+\cos\theta)} (3r \sin \theta) r dr d\theta \\ = \int_0^{2\pi} a^3 (1+\cos\theta)^3 (\sin \theta) d\theta = \left[-\frac{a^3}{4} (1+\cos\theta)^4 \right]_0^{2\pi} = -4a^3 - (-4a^3) = 0$$

$$14. M = y + e^x \ln y, N = \frac{e^x}{y} \Rightarrow \frac{\partial M}{\partial y} = 1 + \frac{e^x}{y}, \frac{\partial N}{\partial x} = \frac{e^x}{y} \Rightarrow \text{Circ} = \iint_R \left[\frac{e^x}{y} - \left(1 + \frac{e^x}{y} \right) \right] dx dy = \iint_R (-1) dx dy \\ = \int_{-1}^1 \int_{x^4+1}^{3-x^2} - dy dx = - \int_{-1}^1 [(3-x^2) - (x^4+1)] dx = \int_{-1}^1 (x^4+x^2-2) dx = -\frac{23}{15}$$

$$15. M = 2xy^3, N = 4x^2y^2 \Rightarrow \frac{\partial M}{\partial y} = 6xy^2, \frac{\partial N}{\partial x} = 8xy^2 \Rightarrow \text{work} = \oint_C 2xy^3 dx + 4x^2y^2 dy = \iint_R (8xy^2 - 6xy^2) dx dy \\ = \int_0^1 \int_0^{x^3} 2xy^2 dy dx = \int_0^1 \frac{2}{3} x^{10} dx = \frac{2}{33}$$

$$16. M = 4x - 2y, N = 2x - 4y \Rightarrow \frac{\partial M}{\partial y} = -2, \frac{\partial N}{\partial x} = 2 \Rightarrow \text{work} = \oint_C (4x - 2y) dx + (2x - 4y) dy \\ = \iint_R [2 - (-2)] dx dy = 4 \iint_R dx dy = 4(\text{Area of the circle}) = 4(\pi \cdot 4) = 16\pi$$

$$17. M = y^2, N = x^2 \Rightarrow \frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = 2x \Rightarrow \oint_C y^2 dx + x^2 dy = \iint_R (2x - 2y) dy dx$$

$$= \int_0^1 \int_0^{1-x} (2x - 2y) dy dx = \int_0^1 (-3x^2 + 4x - 1) dx = [-x^3 + 2x^2 - x]_0^1 = -1 + 2 - 1 = 0$$

$$18. M = 3y, N = 2x \Rightarrow \frac{\partial M}{\partial y} = 3, \frac{\partial N}{\partial x} = 2 \Rightarrow \oint_C 3y dx + 2x dy = \iint_R (2 - 3) dx dy = \int_0^x \int_0^{\sin x} -1 dy dx \\ = - \int_0^{\pi} \sin x dx = -2$$

$$19. M = 6y + x, N = y + 2x \Rightarrow \frac{\partial M}{\partial y} = 6, \frac{\partial N}{\partial x} = 2 \Rightarrow \oint_C (6y + x) dx + (y + 2x) dy = \iint_R (2 - 6) dy dx \\ = -4(\text{Area of the circle}) = -16\pi$$

$$20. M = 2x + y^2, N = 2xy + 3y \Rightarrow \frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = 2y \Rightarrow \oint_C (2x + y^2) dx + (2xy + 3y) dy = \iint_R (2y - 2y) dx dy = 0$$

$$21. M = x = a \cos t, N = y = a \sin t \Rightarrow dx = -a \sin t dt, dy = a \cos t dt \Rightarrow \text{Area} = \frac{1}{2} \oint_C x dy - y dx \\ = \frac{1}{2} \int_0^{2\pi} (a^2 \cos^2 t + a^2 \sin^2 t) dt = \frac{1}{2} \int_0^{2\pi} a^2 dt = \pi a^2$$

$$22. M = x = a \cos t, N = y = b \sin t \Rightarrow dx = -a \sin t dt, dy = b \cos t dt \Rightarrow \text{Area} = \frac{1}{2} \oint_C x dy - y dx \\ = \frac{1}{2} \int_0^{2\pi} (ab \cos^2 t + ab \sin^2 t) dt = \frac{1}{2} \int_0^{2\pi} ab dt = \pi ab$$

$$23. M = x = a \cos^3 t, N = y = \sin^3 t \Rightarrow dx = -3 \cos^2 t \sin t dt, dy = 3 \sin^2 t \cos t dt \Rightarrow \text{Area} = \frac{1}{2} \oint_C x dy - y dx \\ = \frac{1}{2} \int_0^{2\pi} (3 \sin^2 t \cos^2 t)(\cos^2 t + \sin^2 t) dt = \frac{1}{2} \int_0^{2\pi} (3 \sin^2 t \cos^2 t) dt = \frac{3}{8} \int_0^{2\pi} \sin^2 2t dt = \frac{3}{16} \int_0^{4\pi} \sin^2 u du \\ = \frac{3}{16} \left[\frac{u}{2} - \frac{\sin 2u}{4} \right]_0^{4\pi} = \frac{3}{8}\pi$$

$$24. M = x = t^2, N = y = \frac{t^3}{3} - t \Rightarrow dx = 2t dt, dy = (t^2 - 1) dt \Rightarrow \text{Area} = \frac{1}{2} \oint_C x dy - y dx \\ = \frac{1}{2} \int_{-\sqrt{3}}^{\sqrt{3}} \left[t^2(t^2 - 1) - \left(\frac{t^3}{3} - t \right)(2t) \right] dt = \frac{1}{2} \int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{1}{3}t^4 + t^2 \right) dt = \frac{1}{2} \left[\frac{1}{15}t^5 + \frac{1}{3}t^3 \right]_{-\sqrt{3}}^{\sqrt{3}} = \frac{1}{15}(9\sqrt{3} + 15\sqrt{3})$$

$$= \frac{8}{5}\sqrt{3}$$

25. (a) $M = f(x)$, $N = g(y) \Rightarrow \frac{\partial M}{\partial y} = 0$, $\frac{\partial N}{\partial x} = 0 \Rightarrow \oint_C f(x) dx + g(y) dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$
 $= \iint_R 0 dx dy = 0$

(b) $M = ky$, $N = hx \Rightarrow \frac{\partial M}{\partial y} = k$, $\frac{\partial N}{\partial x} = h \Rightarrow \oint_C ky dx + hx dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$
 $= \iint_R (h - k) dx dy = (h - k)(\text{Area of the region})$

26. $M = xy^2$, $N = x^2y + 2x \Rightarrow \frac{\partial M}{\partial y} = 2xy$, $\frac{\partial N}{\partial x} = 2xy + 2 \Rightarrow \oint_C xy^2 dx + (x^2y + 2x) dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$
 $= \iint_R (2xy + 2 - 2xy) dx dy = 2 \iint_R dx dy = 2 \times \text{Area of the square}$

27. The integral is 0 for any simple closed plane curve C . The reasoning: By the tangential form of Green's

Theorem, with $M = 4x^3y$ and $N = x^4$, $\oint_C 4x^3y x + x^4 dy = \iint_R \left[\underbrace{\frac{\partial}{\partial x}(x^4)}_0 - \underbrace{\frac{\partial}{\partial y}(4x^3y)}_0 \right] dx dy$
 $= \iint_R (4x^3 - 4x^3) dx dy = 0$.

28. The integral is 0 for any simple closed curve C . The reasoning: By the normal form of Green's theorem, with

$$M = -y^3 \text{ and } N = x^3, \oint_C -y^3 dx + x^3 dy = \iint_R \left[\underbrace{\frac{\partial}{\partial x}(-y^3)}_0 + \underbrace{\frac{\partial}{\partial y}(x^3)}_0 \right] dx dy = 0.$$

29. Let $M = x$ and $N = 0 \Rightarrow \frac{\partial M}{\partial x} = 1$ and $\frac{\partial N}{\partial y} = 0 \Rightarrow \oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy \Rightarrow \oint_C x dy$
 $= \iint_R (1 + 0) dx dy \Rightarrow \text{Area of } R = \iint_R dx dy = \oint_C x dy; \text{ similarly, } M = y \text{ and } N = 0 \Rightarrow \frac{\partial M}{\partial y} = 1 \text{ and}$
 $\frac{\partial N}{\partial x} = 0 \Rightarrow \oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y} \right) dy dx \Rightarrow \oint_C y dx = \iint_R (0 - 1) dy dx \Rightarrow - \oint_C y dx$
 $= \iint_R dx dy = \text{Area of } R$

30. $\int_a^b f(x) dx = \text{Area of } R = - \oint_C y dx$, from Exercise 29

$$31. \text{ Let } \delta(x, y) = 1 \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{\int_R \int x \delta(x, y) dA}{\int_R \int \delta(x, y) dA} = \frac{\int_R \int x dA}{\int_R \int dA} = \frac{\int_R \int x dA}{A} \Rightarrow A\bar{x} = \int_R \int x dA = \int_R \int (x + 0) dx dy$$

$$= \oint_C \frac{x^2}{2} dy, A\bar{x} = \int_R x dA = \int_R \int (0 + x) dx dy = - \oint_C xy dx, \text{ and } A\bar{x} = \int_R \int x dA = \int_R \int \left(\frac{2}{3}x + \frac{1}{3}x\right) dx dy \\ = \oint_C \frac{1}{3}x^2 dy - \frac{1}{3}xy dx \Rightarrow \frac{1}{2} \oint_C x^2 dy = - \oint_C xy dx = \frac{1}{3} \oint_C x^2 dy - xy dx = A\bar{x}$$

$$32. \text{ If } \delta(x, y) = 1, \text{ then } I_y = \int_R \int x^2 \delta(x, y) dA = \int_R \int x^2 dA = \int_R \int (x^2 + 0) dy dx = \frac{1}{3} \oint_C x^3 dy,$$

$$\int_R \int x^2 dA = \int_R \int (0 + x^2) dy dx = - \oint_C x^2 y dx, \text{ and } \int_R \int x^2 dA = \int_R \int \left(\frac{3}{4}x^2 + \frac{1}{4}x^2\right) dy dx \\ = \oint_C \frac{1}{4}x^3 dy - \frac{1}{4}x^2 y dx = \frac{1}{4} \oint_C x^3 dy - x^2 y dx \Rightarrow \frac{1}{3} \oint_C x^3 dy = - \oint_C x^2 y dx = \frac{1}{4} \oint_C x^3 dy - x^2 y dx = I_y$$

$$33. M = \frac{\partial f}{\partial y}, N = -\frac{\partial f}{\partial x} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y^2}, \frac{\partial N}{\partial x} = -\frac{\partial^2 f}{\partial x^2} \Rightarrow \oint_C \frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial x} dy = \int_R \int \left(-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right) dx dy = 0 \text{ for such curves } C$$

$$34. M = \frac{1}{4}x^2y + \frac{1}{3}y^3, N = x \Rightarrow \frac{\partial M}{\partial y} = \frac{1}{4}x^2 + y^2, \frac{\partial N}{\partial x} = 1 \Rightarrow \text{Curl} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - \left(\frac{1}{4}x^2 + y^2\right) > 0 \text{ in the interior of the ellipse } \frac{1}{4}x^2 + y^2 = 1 \Rightarrow \text{work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_R \int \left(1 - \frac{1}{4}x^2 - y^2\right) dx dy \text{ will be maximized on the region } R = \{(x, y) | \text{curl } \mathbf{F} \geq 0 \text{ or over the region enclosed by } 1 = \frac{1}{4}x^2 + y^2\}$$

$$35. \text{ (a) } \nabla f = \left(\frac{2x}{x^2+y^2}\right)\mathbf{i} + \left(\frac{2y}{x^2+y^2}\right)\mathbf{j} \Rightarrow M = \frac{2x}{x^2+y^2}, N = \frac{2y}{x^2+y^2}; \text{ since } M, N \text{ are discontinuous at } (0, 0), \text{ we cannot apply Green's Theorem over } C. \text{ Thus, let } C_h \text{ be the circle } x = h \cos \theta, y = h \sin \theta, 0 < h \leq a \text{ and let } C_1 \text{ be the circle } x = a \cos t, y = a \sin t, a > 0. \text{ Then } \oint_C \nabla f \cdot \mathbf{n} ds = \oint_{C_1} M dy - N dx + \oint_{C_h} M dy - N dx \\ = \oint_{C_1} \frac{2x}{x^2+y^2} dx - \frac{2y}{x^2+y^2} dx + \oint_{C_h} \frac{2x}{x^2+y^2} dy - \frac{2y}{x^2+y^2} dx. \text{ In the first integral, let } x = a \cos t, y = a \sin t \Rightarrow dx = -a \sin t dt, dy = a \cos t dt, M = 2a \cos t, N = 2a \sin t, 0 \leq t \leq 2\pi. \text{ In the second integral, let } x = h \cos \theta, y = h \sin \theta \Rightarrow dx = -h \sin \theta d\theta, dy = h \cos \theta d\theta, M = 2h \cos \theta, N = 2h \sin \theta, 0 \leq \theta \leq 2\pi. \text{ Then } \oint_C \nabla f \cdot \mathbf{n} ds = \oint_{C_1} \frac{2x}{x^2+y^2} dy - \frac{2y}{x^2+y^2} dx + \oint_{C_h} \frac{2x}{x^2+y^2} dy - \frac{2y}{x^2+y^2} dx$$

$$\begin{aligned}
&= \oint_{C_1} \frac{(2a \cos t)(a \cos t) dt}{a^2} - \frac{(2a \sin t)(-a \sin t) dt}{a^2} + \oint_{C_h} \frac{(2h \cos \theta)(h \cos \theta) d\theta}{h^2} - \frac{(2h \sin \theta)(-h \sin \theta) d\theta}{h^2} \\
&= \int_0^{2\pi} 2 dt + \int_{2\pi}^0 2 d\theta = 0 \text{ for every } h
\end{aligned}$$

(b) If K is any simple closed curve surrounding C_h (K contains $(0,0)$), then $\oint_C \nabla f \cdot n ds$

$$\begin{aligned}
&= \oint_{C_1} M dy - N dx + \oint_{C_h} M dy - N dx, \text{ and in polar coordinates, } \nabla f \cdot n = M dy - N dx \\
&= \left(\frac{2r \cos \theta}{r^2} \right) (r \cos \theta d\theta + \sin \theta dr) - \left(\frac{2r \sin \theta}{r^2} \right) (-r \sin \theta d\theta + \cos \theta dr) = \frac{2r^2}{r^2} d\theta = 2 d\theta. \text{ Now,}
\end{aligned}$$

$$2\theta \text{ increases by } 4\pi \text{ as } K \text{ is traversed once counterclockwise from } \theta = 0 \text{ to } \theta = 2\pi \Rightarrow \oint_C \nabla f \cdot n ds = 0$$

(since $\oint_{C_h} M dy - N dx = -4\pi$) when $(0,0)$ is in the region, but $\oint_K \nabla f \cdot n ds = 4\pi$ when $(0,0)$ is not in the region.

36. Assume a particle has a closed trajectory in R and let C_1 be the path $\Rightarrow C_1$ encloses a simply connected region $R_1 \Rightarrow C_1$ is a simple closed curve. Then the flux over R_1 is $\oint_{C_1} F \cdot n ds = 0$, since the velocity vectors F are

tangent to C_1 . But $0 = \oint_{C_1} F \cdot n ds = \oint_{C_1} M dy - N dx = \int_{R_1} \int \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy \Rightarrow M_x + N_y = 0$, which is a contradiction. Therefore, C_1 cannot be a closed trajectory.

$$\begin{aligned}
37. &\int_{g_1(y)}^{g_2(y)} \frac{\partial N}{\partial x} dx dy = N(g_2(y), y) - N(g_1(y), y) \Rightarrow \int_c^d \int_{g_1(y)}^{g_2(y)} \left(\frac{\partial N}{\partial x} dx \right) dy = \int_c^d [N(g_2(y), y) - N(g_1(y), y)] dy \\
&= \int_c^d N(g_2(y), y) dy - \int_c^d N(g_1(y), y) dy = \int_c^d N(g_2(y), y) dy + \int_d^c N(g_1(y), y) dy = \int_{C_2} N dy + \int_{C_1} N dy \\
&= \oint_C N dy \Rightarrow \oint_C N dy = \int_R \int \frac{\partial N}{\partial x} dx dy
\end{aligned}$$

$$38. \int_a^b \int_c^d \frac{\partial M}{\partial y} dy dx = \int_a^b [M(x, d) - M(x, c)] dx = \int_a^b M(x, d) dx + \int_b^a M(x, c) dx = - \int_{C_2} M dx - \int_{C_1} M dx.$$

Because x is constant along C_2 and C_4 , $\int_{C_2} M dx = \int_{C_4} M dx = 0$

$$\Rightarrow - \left(\int_{C_1} M dx + \int_{C_2} M dx + \int_{C_3} M dx + \int_{C_4} M dx \right) = - \oint_C M dx \Rightarrow \int_a^b \int_c^d \frac{\partial M}{\partial y} dy dx = - \oint_C M dx.$$

39. The curl of a conservative two-dimensional field is zero. The reasoning: A two-dimensional field $\mathbf{F} = Mi + Nj$ can be considered to be the restriction to the xy -plane of a three-dimensional field whose k component is zero, and whose i and j components are independent of z . For such a field to be conservative, we must have

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \text{ by the component test in Section 14.3} \Rightarrow \operatorname{curl} \mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

40. Green's theorem tells us that the circulation of a conservative two-dimensional field around any simple closed curve in the xy -plane is zero. The reasoning: For a conservative field $\mathbf{F} = Mi + Nj$, we have $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ (component test for conservative fields, Section 14.3, Eq. (3)), so $\operatorname{curl} \mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$. By Green's theorem, the counterclockwise circulation around a simple closed plane curve C must equal the integral of $\operatorname{curl} \mathbf{F}$ over the region R enclosed by C . Since $\operatorname{curl} \mathbf{F} = 0$, the latter integral is zero and, therefore, so is the circulation.

The circulation $\oint_C \mathbf{F} \cdot \mathbf{T} ds$ is the same as the work $\oint_C \mathbf{F} \cdot d\mathbf{r}$ done by \mathbf{F} around C , so our observation that circulation of a conservative two-dimensional field is zero agrees with the fact that the work done by a conservative field around a closed curve is always 0.

41-44. Example CAS commands:

Maple:

```
with(plots);
implicitplot({y=4-2*x}, x = 0..3, y = 0..5, scaling=CONSTRAINED);
M:=(x,y) -> x*exp(y);
N:=(x,y) -> 4*x^2*ln(y);
My:=diff(M(x,y),y);
Nx:=diff(N(x,y),x);
int(int(Nx-My, y = 0..4-2*x), x= 0..2);
evalf(%);
```

Mathematica:

```
<< Graphics`ImplicitPlot`
SetOptions[ ImplicitPlot, AspectRatio -> Automatic ];
Clear[x,y]
f[x_,y_] = {
  x E^y ,
  4 x^2 Log[y] }
y1 = 0
y2 = -2 x + 4
Plot[ {y1,y2}, {x,0,2}, AspectRatio -> Automatic ]
integrand = D[f[x,y][[2]],x] - D[f[x,y][[1]],y]
Solve[ c, y ]
{y1,y2} = y /. %
Integrate[ integrand, {x,0,2}, {y,y1,y2} ]
Simplify[%]
N[%]
```

13.5 SURFACE AREA AND SURFACE INTEGRALS

1. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2xi + 2yj - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{(2x)^2 + (2y)^2 + (-1)^2} = \sqrt{4x^2 + 4y^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 1$;

$$\begin{aligned} z = 2 \Rightarrow x^2 + y^2 = 2; \text{ thus } S &= \int_R \int \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \int_R \int \sqrt{4x^2 + 4y^2 + 1} dx dy \\ &= \int_R \int \sqrt{4r^2 \cos^2 \theta + 4r^2 \sin^2 \theta + 1} r dr d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1} r dr d\theta = \int_0^{2\pi} \left[\frac{1}{12} (4r^2 + 1)^{3/2} \right]_0^{\sqrt{2}} d\theta \\ &= \int_0^{2\pi} \frac{13}{6} d\theta = \frac{13}{3}\pi \end{aligned}$$

2. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2xi + 2yj - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 1$; $2 \leq x^2 + y^2 \leq 6$

$$\begin{aligned} \Rightarrow S &= \int_R \int \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \int_R \int \sqrt{4x^2 + 4y^2 + 1} dx dy = \int_R \int \sqrt{4r^2 + 1} r dr d\theta = \int_0^{2\pi} \int_{\sqrt{2}}^{\sqrt{6}} \sqrt{4r^2 + 1} r dr d\theta \\ &= \int_0^{2\pi} \left[\frac{1}{12} (4r^2 + 1)^{3/2} \right]_{\sqrt{2}}^{\sqrt{6}} d\theta = \int_0^{2\pi} \frac{49}{6} d\theta = \frac{49}{3}\pi \end{aligned}$$

3. $\mathbf{p} = \mathbf{k}$, $\nabla f = i + 2j + 2k \Rightarrow |\nabla f| = 3$ and $|\nabla f \cdot \mathbf{p}| = 2$; $x = y^2$ and $x = 2 - y^2$ intersect at $(1, 1)$ and $(1, -1)$

$$\Rightarrow S = \int_R \int \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \int_R \int \frac{3}{2} dx dy = \int_{-1}^1 \int_{y^2}^{2-y^2} \frac{3}{2} dx dy = \int_{-1}^1 (3 - 3y^2) dy = 4$$

4. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2xi - 2k \Rightarrow |\nabla f| = \sqrt{4x^2 + 4} = 2\sqrt{x^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 2 \Rightarrow S = \int_R \int \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA$

$$\begin{aligned} &= \int_R \int \frac{2\sqrt{x^2 + 1}}{2} dx dy = \int_0^{\sqrt{3}} \int_0^x \sqrt{x^2 + 1} dy dx = \int_0^{\sqrt{3}} x \sqrt{x^2 + 1} dx = \left[\frac{1}{3} (x^2 + 1)^{3/2} \right]_0^{\sqrt{3}} = \frac{1}{3} (4)^{3/2} - \frac{1}{3} = \frac{7}{3} \end{aligned}$$

5. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2xi - 2j - 2k \Rightarrow |\nabla f| = \sqrt{(2x)^2 + (-2)^2 + (-2)^2} = \sqrt{4x^2 + 8} = 2\sqrt{x^2 + 2}$ and $|\nabla f \cdot \mathbf{p}| = 2$

$$\begin{aligned} \Rightarrow S &= \int_R \int \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \int_R \int \frac{2\sqrt{x^2 + 2}}{2} dx dy = \int_0^2 \int_0^{3x} \sqrt{x^2 + 2} dy dx = \int_0^2 3x \sqrt{x^2 + 2} dx = \left[(x^2 + 2)^{3/2} \right]_0^2 \\ &= 6\sqrt{6} - 2\sqrt{2} \end{aligned}$$

6. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2xi + 2yj + 2zk \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = \sqrt{8} = 2\sqrt{2}$ and $|\nabla f \cdot \mathbf{p}| = 2z; x^2 + y^2 + z^2 = 2$ and

$$\begin{aligned} z &= \sqrt{x^2 + y^2} \Rightarrow x^2 + y^2 = 1; \text{ thus, } S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \frac{2\sqrt{2}}{2z} dA = \sqrt{2} \iint_R \frac{1}{z} dA \\ &= \sqrt{2} \iint_R \frac{1}{\sqrt{2 - (x^2 + y^2)}} dA = \sqrt{2} \int_0^{2\pi} \int_0^1 \frac{r dr d\theta}{\sqrt{2 - r^2}} = \sqrt{2} \int_0^{2\pi} (-1 + \sqrt{2}) d\theta = 2\pi(2 - \sqrt{2}) \end{aligned}$$

7. $\mathbf{p} = \mathbf{k}$, $\nabla f = ci - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{c^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \sqrt{c^2 + 1} dx dy$

$$= \int_0^{2\pi} \int_0^1 \sqrt{c^2 + 1} r dr d\theta = \int_0^{2\pi} \frac{\sqrt{c^2 + 1}}{2} d\theta = \pi\sqrt{c^2 + 1}$$

8. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2xi + 2zj \Rightarrow |\nabla f| = \sqrt{(2x)^2 + (2z)^2} = 2$ and $|\nabla f \cdot \mathbf{p}| = 2z$ for the upper surface, $z \geq 0$

$$\begin{aligned} \Rightarrow S &= \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \frac{2}{2z} dA = \iint_R \frac{1}{\sqrt{1-x^2}} dy dx = 2 \int_{-1/2}^{1/2} \int_0^{1/2} \frac{1}{\sqrt{1-x^2}} dy dx = \int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}} dx \\ &= [\sin^{-1} x]_{-1/2}^{1/2} = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3} \end{aligned}$$

9. $\mathbf{p} = \mathbf{i}$, $\nabla f = \mathbf{i} + 2yj + 2zk \Rightarrow |\nabla f| = \sqrt{1^2 + (2y)^2 + (2z)^2} = \sqrt{1 + 4y^2 + 4z^2}$ and $|\nabla f \cdot \mathbf{p}| = 1; 1 \leq y^2 + z^2 \leq 4$

$$\begin{aligned} \Rightarrow S &= \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \sqrt{1 + 4y^2 + 4z^2} dy dz = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2 \cos^2 \theta + 4r^2 \sin^2 \theta} r dr d\theta \\ &= \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2} r dr d\theta = \int_0^{2\pi} \left[\frac{1}{12}(1 + 4r^2)^{3/2} \right]_1^2 d\theta = \int_0^{2\pi} \frac{1}{12}(17\sqrt{17} - 5\sqrt{5}) d\theta = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5}) \end{aligned}$$

10. $\mathbf{p} = \mathbf{j}$, $\nabla f = 2xi + \mathbf{j} + 2zk \Rightarrow |\nabla f| = \sqrt{4x^2 + 4z^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 1; y = 0$ and $x^2 + y^2 + z^2 = 2 \Rightarrow x^2 + z^2 = 2$;

$$\text{thus, } S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \sqrt{4x^2 + 4z^2 + 1} dx dz = \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1} r dr d\theta = \int_0^{2\pi} \frac{13}{6} d\theta = \frac{13}{3}\pi$$

11. $\mathbf{p} = \mathbf{k}$, $\nabla f = (2x - \frac{2}{x})\mathbf{i} + \sqrt{15}\mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{(2x - \frac{2}{x})^2 + (\sqrt{15})^2 + (-1)^2} = \sqrt{4x^2 + 8 + \frac{4}{x^2}} = \sqrt{(2x + \frac{2}{x})^2}$

$$\begin{aligned} &= 2x + \frac{2}{x}, \text{ on } 1 \leq x \leq 2 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R (2x + 2x^{-1}) dx dy \\ &= \int_0^1 \int_1^2 (2x + 2x^{-1}) dx dy = \int_0^1 [x^2 + 2 \ln x]_1^2 dy = \int_0^1 (3 + 2 \ln 2) dy = 3 + 2 \ln 2 \end{aligned}$$

12. $\mathbf{p} = \mathbf{k}$, $\nabla f = 3\sqrt{x}\mathbf{i} + 3\sqrt{y}\mathbf{j} - 3\mathbf{k} \Rightarrow |\nabla f| = \sqrt{9x+9y+9} = 3\sqrt{x+y+1}$ and $|\nabla f \cdot \mathbf{p}| = 3$

$$\begin{aligned} \Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA &= \iint_R \sqrt{x+y+1} dx dy = \int_0^1 \int_0^1 \sqrt{x+y+1} dx dy = \int_0^1 \left[\frac{2}{3}(x+y+1)^{3/2} \right]_0^1 dy \\ &= \int_0^1 \left[\frac{2}{3}(y+2)^{3/2} - \frac{2}{3}(y+1)^{3/2} \right] dy = \left[\frac{4}{15}(y+2)^{5/2} - \frac{4}{15}(y+1)^{5/2} \right]_0^1 = \frac{4}{15}[(3)^{5/2} - (2)^{5/2} - (2)^{5/2} + 1] \\ &= \frac{4}{15}(9\sqrt{3} - 8\sqrt{2} + 1) \end{aligned}$$

13. The bottom face S of the cube is in the xy -plane $\Rightarrow z = 0 \Rightarrow g(x, y, 0) = x + y$ and $f(x, y, z) = z = 0 \Rightarrow \mathbf{p} = \mathbf{k}$

$$\begin{aligned} \text{and } \nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx dy \Rightarrow \iint_S g d\sigma &= \iint_R (x+y) dx dy \\ &= \int_0^a \int_0^a (x+y) dx dy = \int_0^a \left(\frac{a^2}{2} + ay \right) dy = a^3. \text{ Because of symmetry, we also get } a^3 \text{ over the face of the cube} \end{aligned}$$

in the xz -plane and a^3 over the face of the cube in the yz -plane. Next, on the top of the cube, $g(x, y, z) = g(x, y, a) = x + y + a$ and $f(x, y, z) = z = a \Rightarrow \mathbf{p} = \mathbf{k}$ and $\nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx dy$

$$\iint_S g d\sigma = \iint_R (x+y+a) dx dy = \int_0^a \int_0^a (x+y+a) dx dy = \int_0^a \int_0^a (x+y) dx dy + \int_0^a \int_0^a a dx dy = 2a^3.$$

Because of symmetry, the integral is also $2a^3$ over each of the other two faces. Therefore,

$$\int_{\text{cube}} (x+y+z) d\sigma = 3(a^3 + 2a^3) = 9a^3.$$

14. On the face S in the xz -plane, we have $y = 0 \Rightarrow f(x, y, z) = y = 0$ and $g(x, y, z) = g(x, 0, z) = z \Rightarrow \mathbf{p} = \mathbf{j}$ and

$$\begin{aligned} \nabla f = \mathbf{j} \Rightarrow |\nabla f| = 1 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx dz &\Rightarrow \iint_S g d\sigma = \iint_S (y+z) d\sigma = \int_0^1 \int_0^2 z dx dz = \int_0^1 2z dz \\ &= 1. \end{aligned}$$

On the face in the xy -plane, we have $z = 0 \Rightarrow f(x, y, z) = z = 0$ and $g(x, y, z) = g(x, y, 0) = y \Rightarrow \mathbf{p} = \mathbf{k}$ and

$$\nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx dy \Rightarrow \iint_S g d\sigma = \iint_S y d\sigma = \int_0^1 \int_0^2 y dx dy = 1.$$

On the triangular face in the plane $x = 2$ we have $f(x, y, z) = x = 2$ and $g(x, y, z) = g(2, y, z) = y + z \Rightarrow \mathbf{p} = \mathbf{i}$ and

$$\begin{aligned} \nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1 \text{ and } |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dz dy &\Rightarrow \iint_S g d\sigma = \iint_S (y+z) d\sigma = \int_0^1 \int_0^{1-y} (y+z) dz dy \\ &= \int_0^1 \frac{1}{2}(1-y^2) dy = \frac{1}{3}. \end{aligned}$$

On the triangular face in the yz -plane, we have $x = 0 \Rightarrow f(x, y, z) = x = 0$ and $g(x, y, z) = g(0, y, z) = y + z$

$$\Rightarrow p = i \text{ and } \nabla f = i \Rightarrow |\nabla f| = 1 \text{ and } |\nabla f \cdot p| = 1 \Rightarrow d\sigma = dz dy \Rightarrow \iint_S g \, d\sigma = \iint_S (y + z) \, d\sigma \\ = \int_0^1 \int_0^{1-y} (y + z) \, dz \, dy = \frac{1}{3}.$$

Finally, on the sloped face, we have $y + z = 1 \Rightarrow f(x, y, z) = y + z = 1$ and $g(x, y, z) = y + z = 1 \Rightarrow p = k$ and

$$\nabla f = j + k \Rightarrow |\nabla f| = \sqrt{2} \text{ and } |\nabla f \cdot p| = 1 \Rightarrow d\sigma = \sqrt{2} \, dx \, dy \Rightarrow \iint_S g \, d\sigma = \iint_S (y + z) \, d\sigma \\ = \int_0^1 \int_0^2 \sqrt{2} \, dx \, dy = 2\sqrt{2}. \text{ Therefore, } \iint_{\text{wedge}} g(x, y, z) \, d\sigma = 1 + 1 + \frac{1}{3} + \frac{1}{3} + 2\sqrt{2} = \frac{8}{3} + 2\sqrt{2}$$

15. On the faces in the coordinate planes, $g(x, y, z) = 0 \Rightarrow$ the integral over these faces is 0.

On the face $x = a$, we have $f(x, y, z) = x = a$ and $g(x, y, z) = g(a, y, z) = ayz \Rightarrow p = i$ and $\nabla f = i \Rightarrow |\nabla f| = 1$

$$\text{and } |\nabla f \cdot p| = 1 \Rightarrow d\sigma = dy \, dz \Rightarrow \iint_S g \, d\sigma = \iint_S ayz \, d\sigma = \int_0^c \int_0^b ayz \, dy \, dz = \frac{ab^2c^2}{4}.$$

On the face $y = b$, we have $f(x, y, z) = y = b$ and $g(x, y, z) = g(x, b, z) = bxz \Rightarrow p = j$ and $\nabla f = j \Rightarrow |\nabla f| = 1$

$$\text{and } |\nabla f \cdot p| = 1 \Rightarrow d\sigma = dx \, dz \Rightarrow \iint_S g \, d\sigma = \iint_S bxz \, d\sigma = \int_0^c \int_0^a bxz \, dz \, dx = \frac{a^2bc^2}{4}.$$

On the face $z = c$, we have $f(x, y, z) = z = c$ and $g(x, y, z) = g(x, y, c) = cxy \Rightarrow p = k$ and $\nabla f = k \Rightarrow |\nabla f| = 1$

$$\text{and } |\nabla f \cdot p| = 1 \Rightarrow d\sigma = dy \, dx \Rightarrow \iint_S g \, d\sigma = \iint_S cxy \, d\sigma = \int_0^b \int_0^a cxy \, dx \, dy = \frac{a^2b^2c}{4}. \text{ Therefore,}$$

$$\iint_S g(x, y, z) \, d\sigma = \frac{abc(ab + ac + bc)}{4}.$$

16. On the face $x = a$, we have $f(x, y, z) = x = a$ and $g(x, y, z) = g(a, y, z) = ayz \Rightarrow p = i$ and $\nabla f = i \Rightarrow |\nabla f| = 1$

$$\text{and } |\nabla f \cdot p| = 1 \Rightarrow d\sigma = dz \, dy \Rightarrow \iint_S g \, d\sigma = \iint_S ayz \, d\sigma = \int_{-b}^b \int_{-c}^c ayz \, dz \, dy = 0. \text{ Because of the symmetry}$$

of g on all the other faces, all the integrals are 0, and $\iint_S g(x, y, z) \, d\sigma = 0$.

17. $f(x, y, z) = 2x + 2y + z = 2 \Rightarrow \nabla f = 2i + 2j + k$ and $g(x, y, z) = x + y + (2 - 2x - 2y) = 2 - x - y \Rightarrow p = k$,

$$|\nabla f| = 3 \text{ and } |\nabla f \cdot p| = 1 \Rightarrow d\sigma = 3 \, dy \, dx; z = 0 \Rightarrow 2x + 2y = 2 \Rightarrow y = 1 - x \Rightarrow \iint_S g \, d\sigma = \iint_S (2 - x - y) \, d\sigma$$

$$= 3 \int_0^1 \int_0^{1-x} (2-x-y) dy dx = 3 \int_0^1 \left[(2-x)(1-x) - \frac{1}{2}(1-x)^2 \right] dx = 3 \int_0^1 \left(\frac{3}{2} - 2x + \frac{x^2}{2} \right) dx = 2$$

18. $f(x, y, z) = y^2 + 4z = 16 \Rightarrow \nabla f = 2y\mathbf{j} + 4\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 16} = 2\sqrt{y^2 + 4}$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 4$

$$\begin{aligned} \Rightarrow d\sigma &= \frac{2\sqrt{y^2+4}}{4} dx dy \Rightarrow \iint_S g d\sigma = \int_{-4}^4 \int_0^1 (x\sqrt{y^2+4}) \left(\frac{\sqrt{y^2+4}}{2} \right) dx dy = \int_{-4}^4 \int_0^1 \frac{x(y^2+4)}{2} dx dy \\ &= \int_{-4}^4 \frac{1}{4}(y^2+4) dy = \frac{1}{2} \left[\frac{y^3}{3} + 4y \right]_0^4 = \frac{1}{2} \left(\frac{64}{3} + 16 \right) = \frac{56}{3} \end{aligned}$$

19. $g(x, y, z) = z, \mathbf{p} = \mathbf{k} \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1$ and $|\nabla g \cdot \mathbf{p}| = 1 \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_R (\mathbf{F} \cdot \mathbf{k}) dA$

$$= \int_0^2 \int_0^3 3 dy dx = 18$$

20. $g(x, y, z) = y, \mathbf{p} = -\mathbf{j} \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1$ and $|\nabla g \cdot \mathbf{p}| = 1 \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_R (\mathbf{F} \cdot -\mathbf{j}) dA$

$$= \int_{-1}^2 \int_2^7 2 dz dx = \int_{-1}^2 2(7-2) dx = 10(2+1) = 30$$

21. $\nabla g = 2xi + 2yj + 2zk \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a; \mathbf{n} = \frac{2xi + 2yj + 2zk}{2\sqrt{x^2 + y^2 + z^2}} = \frac{xi + yj + zk}{a} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{z^2}{a};$

$$\begin{aligned} |\nabla g \cdot \mathbf{k}| &= 2z \Rightarrow d\sigma = \frac{2a}{2z} dA \Rightarrow \text{Flux} = \iint_R \left(\frac{z^2}{a} \right) \left(\frac{a}{z} \right) dA = \iint_R z dA = \iint_R \sqrt{a^2 - (x^2 + y^2)} dx dy \\ &= \int_0^{\pi/2} \int_0^a \sqrt{a^2 - r^2} r dr d\theta = \frac{\pi a^3}{6} \end{aligned}$$

22. $\nabla g = 2xi + 2yj + 2zk \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a; \mathbf{n} = \frac{2xi + 2yj + 2zk}{2\sqrt{x^2 + y^2 + z^2}} = \frac{xi + yj + zk}{a} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{-xy}{a} + \frac{xy}{a}$

$$= 0; |\nabla g \cdot \mathbf{k}| = 2z \Rightarrow d\sigma = \frac{2a}{2z} dA \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_S 0 d\sigma = 0$$

23. From Exercise 21, $\mathbf{n} = \frac{xi + yj + zk}{a}$ and $d\sigma = \frac{a}{z} dA \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{xy}{a} - \frac{xy}{a} + \frac{z}{a} = \frac{z}{a} \Rightarrow \text{Flux} = \iint_R \left(\frac{z}{a} \right) \left(\frac{a}{z} \right) dA$

$$= \iint_R 1 dA = \frac{\pi a^2}{4}$$

24. From Exercise 21, $\mathbf{n} = \frac{\mathbf{x}\mathbf{i} + \mathbf{y}\mathbf{j} + \mathbf{z}\mathbf{k}}{a}$ and $d\sigma = \frac{a}{2} dA \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{zx^2}{a} + \frac{zy^2}{a} + \frac{z^3}{a} = z \left(\frac{x^2 + y^2 + z^2}{a} \right) = az$

$$\Rightarrow \text{Flux} = \iint_R (za) \left(\frac{a}{2} \right) dx dy = \iint_R a^2 dx dy = a^2 (\text{Area of } R) = \frac{1}{4} \pi a^4$$

25. From Exercise 21, $\mathbf{n} = \frac{\mathbf{x}\mathbf{i} + \mathbf{y}\mathbf{j} + \mathbf{z}\mathbf{k}}{a}$ and $d\sigma = \frac{a}{2} dA \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{x^2}{a} + \frac{y^2}{a} + \frac{z^2}{a} = a \Rightarrow \text{Flux}$

$$\begin{aligned} &= \iint_R a \left(\frac{a}{2} \right) dA = \iint_R \frac{a^2}{2} dA = \iint_R \frac{a^2}{\sqrt{a^2 - (x^2 + y^2)}} dA = \int_0^{\pi/2} \int_0^a \frac{a^2}{\sqrt{a^2 - r^2}} r dr d\theta \\ &= \int_0^{\pi/2} a^2 [-\sqrt{a^2 - r^2}]_0^a d\theta = \frac{\pi a^3}{2} \end{aligned}$$

26. From Exercise 21, $\mathbf{n} = \frac{\mathbf{x}\mathbf{i} + \mathbf{y}\mathbf{j} + \mathbf{z}\mathbf{k}}{a}$ and $d\sigma = \frac{a}{2} dA \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{\left(\frac{x^2}{a}\right) + \left(\frac{y^2}{a}\right) + \left(\frac{z^2}{a}\right)}{\sqrt{x^2 + y^2 + z^2}} = \frac{\left(\frac{a^2}{a}\right)}{\sqrt{x^2 + y^2 + z^2}} = 1$

$$\Rightarrow \text{Flux} = \iint_R \frac{a}{2} dx dy = \iint_R \frac{a}{\sqrt{a^2 - (x^2 + y^2)}} dx dy = \int_0^{\pi/2} \int_0^a \frac{a}{\sqrt{a^2 - r^2}} r dr d\theta = \frac{\pi a^2}{2}$$

27. $g(x, y, z) = y^2 + z = 4 \Rightarrow \nabla g = 2y\mathbf{j} + \mathbf{k} \Rightarrow |\nabla g| = \sqrt{4y^2 + 1} \Rightarrow \mathbf{n} = \frac{2y\mathbf{j} + \mathbf{k}}{\sqrt{4y^2 + 1}}$

$$\Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{2xy - 3z}{\sqrt{4y^2 + 1}}; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4y^2 + 1} dA \Rightarrow \text{Flux}$$

$$\begin{aligned} &= \iint_R \left(\frac{2xy - 3z}{\sqrt{4y^2 + 1}} \right) \sqrt{4y^2 + 1} dA = \iint_R (2xy - 3z) dA; z = 0 \text{ and } z = 4 - y^2 \Rightarrow y^2 = 4 \\ &\Rightarrow \text{Flux} = \iint_R [2xy - 3(4 - y^2)] dA = \int_0^1 \int_{-2}^2 (2xy - 12 + 3y^2) dy dx = \int_0^1 [xy^2 - 12y + y^3]_{-2}^2 dx \\ &= \int_0^1 -32 dx = -32 \end{aligned}$$

28. $g(x, y, z) = x^2 + y^2 - z = 0 \Rightarrow \nabla g = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{4(x^2 + y^2) + 1}$

$$\Rightarrow \mathbf{n} = \frac{2xi + 2yj - k}{\sqrt{4(x^2 + y^2) + 1}} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{8x^2 + 8y^2 - 2}{\sqrt{4(x^2 + y^2) + 1}}; \mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4(x^2 + y^2) + 1} dA$$

$$\Rightarrow \text{Flux} = \iint_R \left(\frac{8x^2 + 8y^2 - 2}{\sqrt{4(x^2 + y^2) + 1}} \right) \sqrt{4(x^2 + y^2) + 1} dA = \iint_R (8x^2 + 8y^2 - 2) dA; z = 1 \text{ and } x^2 + y^2 = z$$

$$\Rightarrow x^2 + y^2 = 1 \Rightarrow \text{Flux} = \int_0^{2\pi} \int_0^1 (8r^2 - 2) r dr d\theta = 2\pi$$

29. $g(x, y, z) = y - e^x = 0 \Rightarrow \nabla g = -e^x i + j \Rightarrow |\nabla g| = \sqrt{e^{2x} + 1} \Rightarrow n = \frac{e^x i - j}{\sqrt{e^{2x} + 1}} \Rightarrow F \cdot n = \frac{-2e^x - 2y}{\sqrt{e^{2x} + 1}}; p = i$

$$\Rightarrow |\nabla g \cdot p| = e^x \Rightarrow d\sigma = \frac{\sqrt{e^{2x} + 1}}{e^x} dA \Rightarrow \text{Flux} = \iint_R \left(\frac{-2e^x - 2y}{\sqrt{e^{2x} + 1}} \right) \left(\frac{\sqrt{e^{2x} + 1}}{e^x} \right) dA = \iint_R \frac{-2e^x - 2e^x}{e^x} dA \\ = \iint_R -4 dA = \int_0^1 \int_1^2 -4 dy dz = -4$$

30. $g(x, y, z) = y - \ln x = 0 \Rightarrow \nabla g = -\frac{1}{x} i + j \Rightarrow |\nabla g| = \sqrt{\frac{1}{x^2} + 1} = \frac{\sqrt{1+x^2}}{x} \text{ since } 1 \leq x \leq e$

$$\Rightarrow n = \frac{(-\frac{1}{x} i + j)}{\left(\frac{\sqrt{1+x^2}}{x}\right)} = \frac{-i + xj}{\sqrt{1+x^2}} \Rightarrow F \cdot n = \frac{2xy}{\sqrt{1+x^2}}; p = j \Rightarrow |\nabla g \cdot p| = 1 \Rightarrow d\sigma = \frac{\sqrt{1+x^2}}{x} dA \\ \Rightarrow \text{Flux} = \iint_R \left(\frac{2xy}{\sqrt{1+x^2}} \right) \left(\frac{\sqrt{1+x^2}}{x} \right) dA = \int_0^1 \int_1^e 2y dx dz = \int_1^e \int_0^1 2 \ln x dz dx = \int_1^e 2 \ln x dx \\ = 2[x \ln x - x]_1^e = 2(e - e) - 2(0 - 1) = 2$$

31. On the face $z = a$: $g(x, y, z) = z \Rightarrow \nabla g = k \Rightarrow |\nabla g| = 1; n = k \Rightarrow F \cdot n = 2xz = 2ax$ since $z = a$;

$$d\sigma = dx dy \Rightarrow \text{Flux} = \iint_R 2ax dx dy = \int_0^a \int_0^a 2ax dx dy = a^4.$$

On the face $z = 0$: $g(x, y, z) = z \Rightarrow \nabla g = k \Rightarrow |\nabla g| = 1; n = -k \Rightarrow F \cdot n = -2xz = 0$ since $z = 0$;

$$d\sigma = dx dy \Rightarrow \text{Flux} = \iint_R 0 dx dy = 0.$$

On the face $x = a$: $g(x, y, z) = x \Rightarrow \nabla g = i \Rightarrow |\nabla g| = 1; n = i \Rightarrow F \cdot n = 2xy = 2ay$ since $x = a$;

$$d\sigma = dy dz \Rightarrow \text{Flux} = \int_0^a \int_0^a 2ay dy dz = a^4.$$

On the face $x = 0$: $g(x, y, z) = x \Rightarrow \nabla g = i \Rightarrow |\nabla g| = 1; n = -i \Rightarrow F \cdot n = -2xy = 0$ since $x = 0$

$$\Rightarrow \text{Flux} = 0.$$

On the face $y = a$: $g(x, y, z) = y \Rightarrow \nabla g = j \Rightarrow |\nabla g| = 1; n = j \Rightarrow F \cdot n = 2yz = 2az$ since $y = a$;

$$d\sigma = dz dx \Rightarrow \text{Flux} = \int_0^a \int_0^a 2az dz dx = a^4.$$

On the face $y = 0$: $g(x, y, z) = y \Rightarrow \nabla g = j \Rightarrow |\nabla g| = 1; n = -j \Rightarrow F \cdot n = -2yz = 0$ since $y = 0$

$$\Rightarrow \text{Flux} = 0. \text{ Therefore, Total Flux} = 3a^4.$$