

$$= \frac{1}{2} \int_{-2}^2 (2-x) \left[\frac{4-x^2}{4} - \frac{4-x^2}{4} \right] dx = 0 \Rightarrow \bar{x} = -\frac{1}{2} \text{ and } \bar{y} = 0$$

$$(b) M_{xy} = \int_{-2}^2 \int_{(-\sqrt{4-x^2})/2}^{(\sqrt{4-x^2})/2} z dz dy dx = \frac{1}{2} \int_{-2}^2 \int_{(-\sqrt{4-x^2})/2}^{(\sqrt{4-x^2})/2} (2-x)^2 dy dx = \frac{1}{2} \int_{-2}^2 (2-x)^2 (\sqrt{4-x^2}) dx \\ = 5\pi \Rightarrow \bar{z} = \frac{5}{4}$$

$$7. (a) M = 4 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^4 dz dy dx = 4 \int_0^{\pi/2} \int_0^2 \int_{r^2}^4 r dz dr d\theta = 4 \int_0^{\pi/2} \int_0^2 (4r - r^3) dr d\theta = 4 \int_0^{\pi/2} 4 d\theta = 8\pi;$$

$$M_{xy} = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 zr dz dr d\theta = \int_0^{2\pi} \int_0^2 \frac{r}{2} (16 - r^4) dr d\theta = \frac{32}{3} \int_0^{2\pi} d\theta = \frac{64\pi}{3} \Rightarrow \bar{z} = \frac{8}{3}, \text{ and } \bar{x} = \bar{y} = 0,$$

by symmetry

$$(b) M = 8\pi \Rightarrow 4\pi = \int_0^{2\pi} \int_0^{\sqrt{c}} \int_{r^2}^c r dz dr d\theta = \int_0^{2\pi} \int_0^{\sqrt{c}} (cr - r^3) dr d\theta = \int_0^{2\pi} \frac{c^2}{4} d\theta = \frac{c^2\pi}{2} \Rightarrow c^2 = 8 \Rightarrow c = 2\sqrt{2},$$

since $c > 0$

$$8. M = 8; M_{xy} = \int_{-1}^1 \int_3^5 \int_{-1}^1 z dz dy dx = \int_{-1}^1 \int_3^5 \left[\frac{z^2}{2} \right]_{-1}^1 dy dx = 0; M_{yz} = \int_{-1}^1 \int_3^5 \int_{-1}^1 x dz dy dx \\ = 2 \int_{-1}^1 \int_3^5 x dy dx = 4 \int_{-1}^1 x^2 dx = 0; M_{xz} = \int_{-1}^1 \int_3^5 \int_{-1}^1 y dz dy dx = 2 \int_{-1}^1 \int_3^5 y dy dx = 16 \int_{-1}^1 dx = 32 \\ \Rightarrow \bar{x} = 0, \bar{y} = 4, \bar{z} = 0; I_x = \int_{-1}^1 \int_3^5 \int_{-1}^1 (y^2 + z^2) dz dy dx = \int_{-1}^1 \int_3^5 \left(2y^2 + \frac{2}{3} \right) dy dx = \frac{2}{3} \int_{-1}^1 100 dx = \frac{400}{3};$$

$$I_y = \int_{-1}^1 \int_3^5 \int_{-1}^1 (x^2 + z^2) dz dy dx = \int_{-1}^1 \int_3^5 \left(2x^2 + \frac{2}{3} \right) dy dx = \frac{4}{3} \int_{-1}^1 (3x^2 + 1) dx = \frac{16}{3};$$

$$I_z = \int_{-1}^1 \int_3^5 \int_{-1}^1 (x^2 + y^2) dz dy dx = 2 \int_{-1}^1 \int_3^5 (x^2 + y^2) dy dx = 2 \int_{-1}^1 \left(2x^2 + \frac{98}{3} \right) dx = \frac{400}{3} \Rightarrow R_x = R_z = \sqrt{\frac{50}{3}}$$

$$\text{and } R_y = \sqrt{\frac{7}{3}}$$

$$9. \text{ The plane } y + 2z = 2 \text{ is the top of the wedge } \Rightarrow I_L = \int_{-2}^2 \int_{-2}^4 \int_{-1}^{(2-y)/2} [(y-6)^2 + z^2] dz dy dx \\ = \int_{-2}^2 \int_{-2}^4 \left[\frac{(y-6)^2(4-y)}{2} + \frac{(2-y)^3}{24} + \frac{1}{3} \right] dy dx; \text{ let } t = 2-y \Rightarrow I_L = 4 \int_{-2}^4 \left(\frac{13t^3}{24} + 5t^2 + 16t + \frac{49}{3} \right) dt = 1386;$$

$$M = \frac{1}{2}(3)(6)(4) = 36 \Rightarrow R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{77}{2}}$$

10. The plane $y + 2z = 2$ is the top of the wedge $\Rightarrow I_L = \int_{-2}^2 \int_{-2}^4 \int_{-1}^{(2-y)/2} [(x-4)^2 + y^2] dz dy dx$

$$= \frac{1}{2} \int_{-2}^2 \int_{-2}^4 (x^2 - 8x + 16 + y^2)(4-y) dy dx = \int_{-2}^2 (9x^2 - 72x + 162) dx = 696; M = \frac{1}{2}(3)(6)(4) = 36$$

$$\Rightarrow R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{58}{3}}$$

11. $M = 8; I_L = \int_0^4 \int_0^2 \int_0^1 [z^2 + (y-2)^2] dz dy dx = \int_0^4 \int_0^2 \left(y^2 - 4y + \frac{13}{3}\right) dy dx = \frac{10}{3} \int_0^4 dx = \frac{40}{3}$

$$\Rightarrow R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{5}{3}}$$

12. $M = 8; I_L = \int_0^4 \int_0^2 \int_0^1 [(x-4)^2 + y^2] dz dy dx = \int_0^4 \int_0^2 [(x-4)^2 + y^2] dy dx = \int_0^4 \left[2(x-4)^2 + \frac{8}{3}\right] dx = \frac{160}{3}$

$$\Rightarrow R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{20}{3}}$$

13. (a) $M = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} 2x dz dy dx = \int_0^2 \int_0^{2-x} (4x - 2x^2 - 2xy) dy dx = \int_0^2 (x^3 - 4x^2 + 4x) dx = \frac{4}{3}$

(b) $M_{xy} = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} 2xz dz dy dx = \int_0^2 \int_0^{2-x} x(2-x-y)^2 dy dx = \int_0^2 \frac{x(2-x)^3}{3} dx = \frac{8}{15}; M_{xz} = \frac{8}{15}$ by symmetry; $M_{yz} = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} 2x^2 dz dy dx = \int_0^2 \int_0^{2-x} 2x^2(2-x-y) dy dx = \int_0^2 (2x-x^2)^2 dx = \frac{16}{15}$

$$\Rightarrow \bar{x} = \frac{4}{5}, \text{ and } \bar{y} = \bar{z} = \frac{2}{5}$$

14. (a) $M = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kxy dz dy dx = k \int_0^2 \int_0^{\sqrt{x}} xy(4-x^2) dy dx = \frac{k}{2} \int_0^2 (4x^2 - x^4) dx = \frac{32k}{15}$

(b) $M_{yz} = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kx^2y dz dy dx = k \int_0^2 \int_0^{\sqrt{x}} x^2y(4-x^2) dy dx = \frac{k}{2} \int_0^2 (4x^3 - x^5) dx = \frac{8k}{3}$

$$\Rightarrow \bar{x} = \frac{5}{4}; M_{xz} = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kx^2y^2 dz dy dx = k \int_0^2 \int_0^{\sqrt{x}} xy^2(4-x^2) dy dx = \frac{k}{3} \int_0^2 (4x^{5/2} - x^{9/2}) dx$$

$$\begin{aligned}
&= \frac{256\sqrt{2}k}{231} \Rightarrow \bar{y} = \frac{40\sqrt{2}}{77}; M_{xy} = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kxyz \, dz \, dy \, dx = \frac{k}{2} \int_0^2 \int_0^{\sqrt{x}} xy(4-x^2)^2 \, dy \, dx \\
&= \frac{k}{4} \int_0^2 (16x^2 - 8x^4 + x^6) \, dx = \frac{256k}{105} \Rightarrow \bar{z} = \frac{8}{7}
\end{aligned}$$

15. (a) $M = \int_0^1 \int_0^1 \int_0^1 (x+y+z+1) \, dz \, dy \, dx = \int_0^1 \int_0^1 \left(x+y+\frac{3}{2}\right) \, dy \, dx = \int_0^1 (x+2) \, dx = \frac{5}{2}$

(b) $M_{xy} = \int_0^1 \int_0^1 \int_0^1 z(x+y+z+1) \, dz \, dy \, dx = \frac{1}{2} \int_0^1 \int_0^1 \left(x+y+\frac{5}{3}\right) \, dy \, dx = \frac{1}{2} \int_0^1 \left(x+\frac{13}{6}\right) \, dx = \frac{4}{3}$

$\Rightarrow M_{xy} = M_{yz} = M_{xz} = \frac{4}{3}$, by symmetry $\Rightarrow \bar{x} = \bar{y} = \bar{z} = \frac{8}{15}$

(c) $I_z = \int_0^1 \int_0^1 \int_0^1 (x^2+y^2)(x+y+z+1) \, dz \, dy \, dx = \int_0^1 \int_0^1 (x^2+y^2)\left(x+y+\frac{3}{2}\right) \, dy \, dx$

 $= \int_0^1 \left(x^3+2x^2+\frac{1}{3}x+\frac{3}{4}\right) \, dx = \frac{11}{6} \Rightarrow I_x = I_y = I_z = \frac{11}{6}$, by symmetry

(d) $R_x = R_y = R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{11}{15}}$

16. The plane $y+2z=2$ is the top of the wedge.

(a) $M = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1) \, dz \, dy \, dx = \int_{-1}^1 \int_{-2}^4 (x+1)\left(2-\frac{y}{2}\right) \, dy \, dx = 18$

(b) $M_{yz} = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} x(x+1) \, dz \, dy \, dx = \int_{-1}^1 \int_{-2}^4 x(x+1)\left(2-\frac{y}{2}\right) \, dy \, dx = 6;$

$M_{xz} = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} y(x+1) \, dz \, dy \, dx = \int_{-1}^1 \int_{-2}^4 y(x+1)\left(2-\frac{y}{2}\right) \, dy \, dx = 0;$

$M_{xy} = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} z(x+1) \, dz \, dy \, dx = \frac{1}{2} \int_{-1}^1 \int_{-2}^4 (x+1)\left(\frac{y^2}{4}-y\right) \, dy \, dx = 0 \Rightarrow \bar{x} = \frac{4}{3}$, and $\bar{y} = \bar{z} = 0$

(c) $I_x = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1)(y^2+z^2) \, dz \, dy \, dx = \int_{-1}^1 \int_{-2}^4 (x+1) \left[2y^2 + \frac{1}{3} - \frac{y^3}{2} + \frac{1}{3}(1-\frac{y}{2})^3\right] \, dy \, dx = 45;$

$I_y = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1)(x^2+z^2) \, dz \, dy \, dx = \int_{-1}^1 \int_{-2}^4 (x+1) \left[2x^2 + \frac{1}{3} - \frac{x^2y}{2} + \frac{1}{3}(1-\frac{y}{2})^3\right] \, dy \, dx = 15;$

$$I_z = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1)(x^2+y^2) dz dy dx = \int_{-1}^1 \int_{-2}^4 (x+1)\left(2-\frac{y}{2}\right)(x^2+y^2) dy dx = 42$$

$$(d) R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{5}{2}}, R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{5}{6}}, \text{ and } R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{7}{3}}$$

$$17. M = \int_0^1 \int_{z-1}^{1-z} \int_0^{\sqrt{z}} (2y+5) dy dx dz = \int_0^1 \int_{z-1}^{1-z} (z+5\sqrt{z}) dx dz = \int_0^1 2(z+5\sqrt{z})(1-z) dz \\ = 2 \int_0^1 (5z^{1/2} + z - 5z^{3/2} - z^2) dz = 2 \left[\frac{10}{3}z^{3/2} + \frac{1}{2}z^2 - 2z^{5/2} - \frac{1}{3}z^3 \right]_0^1 = 2 \left(\frac{9}{3} - \frac{3}{2} \right) = 3$$

$$18. M = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2(x^2+y^2)}^{16-2(x^2+y^2)} \sqrt{x^2+y^2} dz dy dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}[16-4(x^2+y^2)] dy dx \\ = 4 \int_0^{2\pi} \int_0^2 r(4-r^2) r dr d\theta = 4 \int_0^{2\pi} \left[\frac{4r^3}{3} - \frac{r^5}{5} \right]_0^2 d\theta = 4 \int_0^{2\pi} \frac{64}{15} d\theta = \frac{512\pi}{15}$$

19. (a) Let ΔV_i be the volume of the i th piece, and let (x_i, y_i, z_i) be a point in the i th piece. Then the work done by gravity in moving the i th piece to the xy -plane is approximately $W_i = m_i g z_i = (x_i + y_i + z_i + 1)g \Delta V_i z_i$

$$\Rightarrow \text{the total work done is the triple integral } W = \int_0^1 \int_0^1 \int_0^1 (x+y+z+1)gz dz dy dx \\ = g \int_0^1 \int_0^1 \left[\frac{1}{2}xz^2 + \frac{1}{2}yz^2 + \frac{1}{3}z^3 + \frac{1}{2}z^2 \right]_0^1 dy dx = g \int_0^1 \int_0^1 \left(\frac{1}{2}x + \frac{1}{2}y + \frac{5}{6} \right) dy dx = g \int_0^1 \left[\frac{1}{2}xy + \frac{1}{4}y^2 + \frac{5}{6}y \right]_0^1 dx \\ = g \int_0^1 \left(\frac{1}{2}x + \frac{13}{12} \right) dx = g \left[\frac{x^2}{4} + \frac{13}{12}x \right]_0^1 = g \left(\frac{16}{12} \right) = \frac{4}{3}g$$

- (b) From Exercise 15 the center of mass is $(\frac{8}{15}, \frac{8}{15}, \frac{8}{15})$ and the mass of the liquid is $\frac{5}{2} \Rightarrow$ the work done by gravity in moving the center of mass to the xy -plane is $W = mgd = (\frac{5}{2})(g)(\frac{8}{15}) = \frac{4}{3}g$, which is the same as the work done in part (a).

$$20. (a) \text{From Exercise 19(a) we see that the work done is } W = g \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kxyz dz dy dx \\ = k \cdot g \int_0^2 \int_0^{\sqrt{x}} \frac{1}{2}xy(4-x^2)^2 dy dx = \frac{k \cdot g}{4} \int_0^2 x^2(4-x^2)^2 dx = \frac{k \cdot g}{4} \int_0^2 (16x^2 - 8x^4 + x^6) dx \\ = \frac{k \cdot g}{4} \left[\frac{16}{3}x^3 - \frac{8}{5}x^5 + \frac{1}{7}x^7 \right]_0^2 = \frac{256k \cdot g}{105}$$

(b) From Exercise 14 the center of mass is $\left(\frac{5}{4}, \frac{40\sqrt{2}}{77}, \frac{8}{7}\right)$ and the mass of the liquid is $\frac{32k}{15} \Rightarrow$ the work done by gravity in moving the center of mass to the xy -plane is $W = mgd = \left(\frac{32k}{15}\right)(g)\left(\frac{8}{7}\right) = \frac{256k \cdot g}{105}$

$$21. (a) \bar{x} = \frac{M_{yz}}{M} = 0 \Rightarrow \int \int \int_R x \delta(x, y, z) dx dy dz = 0 \Rightarrow M_{yz} = 0$$

$$(b) I_L = \int \int \int_R |\mathbf{v} - h\mathbf{i}|^2 dm = \int \int \int_R |(x - h)\mathbf{i} + y\mathbf{j}|^2 dm = \int \int \int_R (x^2 - 2xh + h^2 + y^2) dm \\ = \int \int \int_R (x^2 + y^2) dm - 2h \int \int \int_R x dm + h^2 \int \int \int_R dm = I_x - 0 + h^2 m = I_{c.m.} + h^2 m$$

$$22. I_L = I_{c.m.} + mh^2 = \frac{2}{5}ma^2 + ma^2 = \frac{7}{5}ma^2$$

$$23. (a) (\bar{x}, \bar{y}, \bar{z}) = \left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right) \Rightarrow I_z = I_{c.m.} + abc \left(\sqrt{\frac{a^2}{4} + \frac{b^2}{4}}\right)^2 \Rightarrow I_{c.m.} = I_z - \frac{abc(a^2 + b^2)}{4} \\ = \frac{abc(a^2 + b^2)}{3} - \frac{abc(a^2 + b^2)}{4} = \frac{abc(a^2 + b^2)}{12}; R_{c.m.} = \sqrt{\frac{I_{c.m.}}{M}} = \sqrt{\frac{a^2 + b^2}{12}}$$

$$(b) I_L = I_{c.m.} + abc \left(\sqrt{\frac{a^2}{4} + \left(\frac{b}{2} - 2b\right)^2}\right)^2 = \frac{abc(a^2 + b^2)}{12} + \frac{abc(a^2 + 9b^2)}{4} = \frac{abc(4a^2 + 28b^2)}{12} \\ = \frac{abc(a^2 + 7b^2)}{3}; R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{a^2 + 7b^2}{3}}$$

$$24. M = \int_{-3}^3 \int_{-2}^4 \int_{-4/3}^{(4-2y)/3} dz dy dx = \int_{-3}^3 \int_{-2}^4 \frac{2}{3}(4-y) dy dx = \int_{-3}^3 \frac{2}{3} \left[4y - \frac{y^2}{2}\right]_{-2}^4 dx = 12 \int_{-3}^3 dx = 72; \\ \bar{x} = \bar{y} = \bar{z} = 0 \text{ from Exercise 2} \Rightarrow I_x = I_{c.m.} + 72(\sqrt{0^2 + 0^2})^2 = I_{c.m.} \Rightarrow I_L = I_{c.m.} + 72 \left(\sqrt{16 + \frac{16}{9}}\right)^2 \\ = 208 + 72 \left(\frac{160}{9}\right) = 1488$$

$$25. M_{yz|B_1 \cup B_2} = \int \int \int_{B_1} x dV_1 + \int \int \int_{B_2} x dV_2 = M_{(yz)_1} + M_{(yz)_2} \Rightarrow \bar{x} = \frac{M_{(yz)_1} + M_{(yz)_2}}{m_1 + m_2}; \text{ similarly,} \\ \bar{y} = \frac{M_{(xz)_1} + M_{(xz)_2}}{m_1 + m_2} \text{ and } \bar{z} = \frac{M_{(xy)_1} + M_{(xy)_2}}{m_1 + m_2} \Rightarrow \mathbf{c} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k} \\ = \frac{1}{m_1 + m_2} \left[(M_{(yz)_1} + M_{(yz)_2})\mathbf{i} + (M_{(xz)_1} + M_{(xz)_2})\mathbf{j} + (M_{(xy)_1} + M_{(xy)_2})\mathbf{k} \right] \\ = \frac{1}{m_1 + m_2} [(m_1 \bar{x}_1 + m_2 \bar{x}_2)\mathbf{i} + (m_1 \bar{y}_1 + m_2 \bar{y}_2)\mathbf{j} + (m_1 \bar{z}_1 + m_2 \bar{z}_2)\mathbf{k}]$$

$$= \frac{1}{m_1 + m_2} [m_1(\bar{x}_1\mathbf{i} + \bar{y}_1\mathbf{j} + \bar{z}_1\mathbf{k}) + m_2(\bar{x}_2\mathbf{i} + \bar{y}_2\mathbf{j} + \bar{z}_2\mathbf{k})] = \frac{m_1\mathbf{c}_1 + m_2\mathbf{c}_2}{m_1 + m_2}$$

26. (a) $\mathbf{c} = \frac{12\left(\frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} + \mathbf{k}\right) + 2\left(\frac{1}{2}\mathbf{i} + 4\mathbf{j} + \frac{1}{2}\mathbf{k}\right)}{12+2} = \frac{\frac{13}{2}\mathbf{i} + 13\mathbf{j} + \frac{13}{2}\mathbf{k}}{7} \Rightarrow \bar{x} = \frac{13}{14}, \bar{y} = \frac{13}{7}, \bar{z} = \frac{13}{14}$

(b) $\mathbf{c} = \frac{12\left(\frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} + \mathbf{k}\right) + 12\left(\frac{1}{2}\mathbf{i} + \frac{11}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}\right)}{12+12} = \frac{2\mathbf{i} + 7\mathbf{j} + \frac{1}{2}\mathbf{k}}{2} \Rightarrow \bar{x} = 1, \bar{y} = \frac{7}{2}, \bar{z} = \frac{1}{4}$

(c) $\mathbf{c} = \frac{2\left(\frac{1}{2}\mathbf{i} + 4\mathbf{j} + \frac{1}{2}\mathbf{k}\right) + 12\left(\frac{1}{2}\mathbf{i} + \frac{11}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}\right)}{2+12} = \frac{13\mathbf{i} + 74\mathbf{j} - 5\mathbf{k}}{14} \Rightarrow \bar{x} = \frac{13}{14}, \bar{y} = \frac{37}{7}, \bar{z} = -\frac{5}{14}$

(d) $\mathbf{c} = \frac{12\left(\frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} + \mathbf{k}\right) + 2\left(\frac{1}{2}\mathbf{i} + 4\mathbf{j} + \frac{1}{2}\mathbf{k}\right) + 12\left(\frac{1}{2}\mathbf{i} + \frac{11}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}\right)}{12+2+12} = \frac{25\mathbf{i} + 92\mathbf{j} + 7\mathbf{k}}{26} \Rightarrow \bar{x} = \frac{25}{26}, \bar{y} = \frac{46}{13}, \bar{z} = \frac{7}{26}$

27. (a) $\mathbf{c} = \frac{\left(\frac{\pi a^2 h}{3}\right)\left(\frac{h}{4}\mathbf{k}\right) + \left(\frac{2\pi a^3}{3}\right)\left(-\frac{3a}{8}\mathbf{k}\right)}{m_1 + m_2} = \frac{\left(\frac{a^2 \pi}{3}\right)\left(\frac{h^2 - 3a^2}{4}\mathbf{k}\right)}{m_1 + m_2}$, where $m_1 = \frac{\pi a^2 h}{3}$ and $m_2 = \frac{2\pi a^3}{3}$; if

$$\frac{h^2 - 3a^2}{4} = 0, \text{ or } h = a\sqrt{3}, \text{ then the centroid is on the common base}$$

(b) See the solution to Exercise 55, Section 12.2, to see that $h = a\sqrt{2}$.

28. $\mathbf{c} = \frac{\left(\frac{s^2 h}{3}\right)\left(\frac{h}{4}\mathbf{k}\right) + s^3\left(-\frac{s}{2}\mathbf{k}\right)}{m_1 + m_2} = \frac{\left(\frac{s^2}{12}\right)[(h^2 - 6s^2)\mathbf{k}]}{m_1 + m_2}$, where $m_1 = \frac{s^2 h}{3}$ and $m_2 = s^3$; if $h^2 - 6s^2 < 0$,

or $h < \sqrt{6}s$, then the centroid is in the base of the pyramid.

12.6 TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

$$1. \int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 [r(2-r^2)^{1/2} - r^2] dr \, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(2-r^2)^{3/2} - \frac{r^3}{3} \right]_0^1 d\theta \\ = \int_0^{2\pi} \left(\frac{2^{3/2}}{3} - \frac{2}{3} \right) d\theta = \frac{4\pi(\sqrt{2}-1)}{3}$$

$$2. \int_0^{2\pi} \int_0^3 \int_{r^2/3}^{\sqrt{18-r^2}} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^3 [r(18-r^2)^{1/2} - \frac{r^3}{3}] dr \, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(18-r^2)^{3/2} - \frac{r^4}{12} \right]_0^3 d\theta \\ = \frac{9\pi(8\sqrt{2}-7)}{2}$$

$$3. \int_0^{2\pi} \int_0^{\theta/2\pi} \int_0^{3+24r^2} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^{\theta/2\pi} (3r + 24r^3) dr \, d\theta = \int_0^{2\pi} \left[\frac{3}{2}r^2 + 6r^4 \right]_0^{\theta/2\pi} d\theta = \frac{3}{2} \int_0^{2\pi} \left(\frac{\theta^2}{4\pi^2} + \frac{4\theta^4}{16\pi^4} \right) d\theta$$

$$= \frac{3}{2} \left[\frac{\theta^3}{12\pi^2} + \frac{\theta^5}{5\pi^4} \right]_0^{2\pi} = \frac{17\pi}{5}$$

$$\begin{aligned} 4. & \int_0^{\pi} \int_0^{\theta/\pi} \int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z \, dz \, r \, dr \, d\theta = \int_0^{\pi} \int_0^{\theta/\pi} \frac{1}{2} [9(4-r^2) - (4-r^2)]r \, dr \, d\theta = 4 \int_0^{\pi} \int_0^{\theta/\pi} (4r - r^3) \, dr \, d\theta \\ & = 4 \int_0^{\pi} \left[2r^2 - \frac{r^4}{4} \right]_0^{\theta/\pi} = 4 \int_0^{\pi} \left(\frac{2\theta^2}{\pi^2} - \frac{\theta^4}{4\pi^4} \right) d\theta = \frac{37\pi}{15} \end{aligned}$$

$$\begin{aligned} 5. & \int_0^{2\pi} \int_0^1 \int_r^{(2-r^2)^{-1/2}} 3 \, dz \, r \, dr \, d\theta = 3 \int_0^{2\pi} \int_0^1 \left[r(2-r^2)^{-1/2} - r^2 \right] dr \, d\theta = 3 \int_0^{2\pi} \left[-(2-r^2)^{1/2} - \frac{r^3}{3} \right]_0^1 d\theta \\ & = 3 \int_0^{2\pi} \left(\sqrt{2} - \frac{4}{3} \right) d\theta = \pi(6\sqrt{2} - 8) \end{aligned}$$

$$6. \int_0^{2\pi} \int_0^1 \int_{-1/2}^{1/2} (r^2 \sin^2 \theta + z^2) \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 \left(r^3 \sin^2 \theta + \frac{r}{12} \right) dr \, d\theta = \int_0^{2\pi} \left(\frac{\sin^2 \theta}{4} + \frac{1}{24} \right) d\theta = \frac{\pi}{3}$$

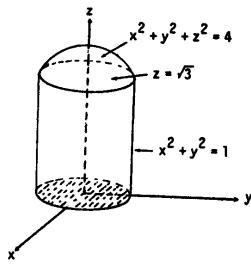
$$7. \int_0^{2\pi} \int_0^3 \int_0^{z/3} r^3 \, dr \, dz \, d\theta = \int_0^{2\pi} \int_0^3 \frac{z^4}{324} \, dz \, d\theta = \int_0^{2\pi} \frac{3}{20} \, d\theta = \frac{3\pi}{10}$$

$$8. \int_{-1}^1 \int_0^{2\pi} \int_0^{1+\cos \theta} 4r \, dr \, d\theta \, dz = \int_{-1}^1 \int_0^{2\pi} 2(1+\cos \theta)^2 \, d\theta \, dz = \int_{-1}^1 6\pi \, d\theta = 12\pi$$

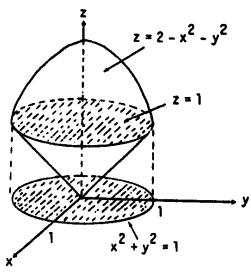
$$\begin{aligned} 9. & \int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2 \cos^2 \theta + z^2) r \, d\theta \, dr \, dz = \int_0^1 \int_0^{\sqrt{z}} \left[\frac{r^2 \theta}{2} + \frac{r^2 \sin 2\theta}{4} + z^2 \theta \right]_0^{2\pi} r \, dr \, dz = \int_0^1 \int_0^{\sqrt{z}} (\pi r^3 + 2\pi r z^2) \, dr \, dz \\ & = \int_0^1 \left[\frac{\pi r^4}{4} + \pi r^2 z^2 \right]_0^{\sqrt{z}} dz = \int_0^1 \left(\frac{\pi z^2}{4} + \pi z^3 \right) dz = \left[\frac{\pi z^3}{12} + \frac{\pi z^4}{4} \right]_0^1 = \frac{\pi}{3} \end{aligned}$$

$$\begin{aligned} 10. & \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \int_0^{2\pi} (r \sin \theta + 1) r \, d\theta \, dz \, dr = \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} 2\pi r \, dz \, dr = 2\pi \int_0^2 \left[r(4-r^2)^{1/2} - r^2 + 2r \right] dr \\ & = 2\pi \left[-\frac{1}{3}(4-r^2)^{3/2} - \frac{r^3}{3} + r^2 \right]_0^2 = 2\pi \left[-\frac{8}{3} + 4 + \frac{1}{3}(4)^{3/2} \right] = 8\pi \end{aligned}$$

11. (a) $\int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} dz \ r \ dr \ d\theta$
 (b) $\int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^1 r \ dr \ dz \ d\theta + \int_0^{2\pi} \int_{\sqrt{3}}^2 \int_0^{\sqrt{4-z^2}} r \ dr \ dz \ d\theta$
 (c) $\int_0^1 \int_0^{\sqrt{4-r^2}} \int_0^{2\pi} r \ d\theta \ dz \ dr$



12. (a) $\int_0^{2\pi} \int_0^1 \int_r^{2-r^2} dz \ r \ dr \ d\theta$
 (b) $\int_0^{2\pi} \int_0^1 \int_0^z r \ dr \ dz \ d\theta + \int_0^{2\pi} \int_1^2 \int_0^{\sqrt{2-z}} r \ dr \ dz \ d\theta$
 (c) $\int_0^1 \int_r^{2-r^2} \int_0^{2\pi} r \ d\theta \ dz \ dr$



13. $\int_{-\pi/2}^{\pi/2} \int_0^{\cos \theta} \int_0^{3r^2} f(r, \theta, z) \ dz \ r \ dr \ d\theta$

14. $\int_{-\pi/2}^{\pi/2} \int_0^1 \int_0^{r \cos \theta} r^3 dz \ dr \ d\theta = \int_{-\pi/2}^{\pi/2} \int_0^1 r^4 \cos \theta dr \ d\theta = \frac{1}{5} \int_{-\pi/2}^{\pi/2} \cos \theta d\theta = \frac{2}{5}$

15. $\int_{-\pi/2}^{\pi/2} \int_0^2 \int_0^{4-r \sin \theta} f(r, \theta, z) \ dz \ r \ dr \ d\theta$

16. $\int_{-\pi/2}^{\pi/2} \int_0^3 \int_0^{5-r \cos \theta} f(r, \theta, z) \ dz \ r \ dr \ d\theta$

17. $\int_{-\pi/2}^{\pi/2} \int_1^{1+\cos \theta} \int_0^4 f(r, \theta, z) \ dz \ r \ dr \ d\theta$

18. $\int_{-\pi/2}^{\pi/2} \int_{\cos \theta}^{2 \cos \theta} \int_0^{3-r \sin \theta} f(r, \theta, z) \ dz \ r \ dr \ d\theta$

19. $\int_0^{\pi/4} \int_0^{\sec \theta} \int_0^{2-r \sin \theta} f(r, \theta, z) \ dz \ r \ dr \ d\theta$

20. $\int_{\pi/4}^{\pi/2} \int_0^{\csc \theta} \int_0^{2-r \sin \theta} f(r, \theta, z) \ dz \ r \ dr \ d\theta$

$$21. \int_0^{\pi} \int_0^{\pi} \int_0^{2\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{8}{3} \int_0^{\pi} \int_0^{\pi} \sin^4\phi \, d\phi \, d\theta = \frac{8}{3} \int_0^{\pi} \left(\left[-\frac{\sin^3\phi \cos\phi}{4} \right]_0^{\pi} + \frac{3}{4} \int_0^{\pi} \sin^2\phi \, d\phi \right) d\theta$$

$$= 2 \int_0^{\pi} \int_0^{\pi} \sin^2\phi \, d\phi \, d\theta = \int_0^{\pi} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi} d\theta = \int_0^{\pi} \pi \, d\theta = \pi^2$$

$$22. \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 (\rho \cos\phi) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/4} 4 \cos\phi \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} [2 \sin^2\phi]_0^{\pi/4} d\theta = \int_0^{2\pi} d\theta = 2\pi$$

$$23. \int_0^{2\pi} \int_0^{\pi} \int_0^{(1-\cos\phi)/2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{1}{24} \int_0^{2\pi} \int_0^{\pi} (1-\cos\phi)^3 \sin\phi \, d\phi \, d\theta = \frac{1}{96} \int_0^{2\pi} [(1-\cos\phi)^4]_0^{\pi} d\theta$$

$$= \frac{1}{96} \int_0^{2\pi} (2^4 - 0) d\theta = \frac{16}{96} \int_0^{2\pi} d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}$$

$$24. \int_0^{3\pi/2} \int_0^{\pi} \int_0^1 5\rho^3 \sin^3\phi \, d\rho \, d\phi \, d\theta = \frac{5}{4} \int_0^{3\pi/2} \int_0^{\pi} \sin^3\phi \, d\phi \, d\theta = \frac{5}{4} \int_0^{3\pi/2} \left(\left[-\frac{\sin^2\phi \cos\phi}{3} \right]_0^{\pi} + \frac{2}{3} \int_0^{\pi} \sin\phi \, d\phi \right) d\theta$$

$$= \frac{5}{6} \int_0^{3\pi/2} [-\cos\phi]_0^{\pi} d\theta = \frac{5}{3} \int_0^{3\pi/2} d\theta = \frac{5\pi}{2}$$

$$25. \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec\phi}^2 3\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} (8 - \sec^3\phi) \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} [-8 \cos\phi - \frac{1}{2} \sec^2\phi]_0^{\pi/3} d\theta$$

$$= \int_0^{2\pi} \left[(-4 - 2) - \left(-8 - \frac{1}{2} \right) \right] d\theta = \frac{5}{2} \int_0^{2\pi} d\theta = 5\pi$$

$$26. \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec\phi} \rho^3 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta = \frac{1}{4} \int_0^{2\pi} \int_0^{\pi/4} \tan\phi \sec^2\phi \, d\phi \, d\theta = \frac{1}{4} \int_0^{2\pi} \left[\frac{1}{2} \tan^2\phi \right]_0^{\pi/4} d\theta$$

$$= \frac{1}{8} \int_0^{2\pi} d\theta = \frac{\pi}{4}$$

$$27. \int_0^2 \int_{-\pi}^0 \int_{\pi/4}^{\pi/2} \rho^3 \sin 2\phi \, d\phi \, d\theta \, d\rho = \int_0^2 \int_{-\pi}^0 \rho^3 \left[-\frac{\cos 2\phi}{2} \right]_{\pi/4}^{\pi/2} d\theta \, d\rho = \int_0^2 \int_{-\pi}^0 \frac{\rho^3}{2} d\theta \, d\rho = \int_0^2 \frac{\rho^3 \pi}{2} d\rho$$

$$= \left[\frac{\pi \rho^4}{8} \right]_0^2 = 2\pi$$

$$\begin{aligned}
28. \int_{\pi/6}^{\pi/3} \int_{\csc \phi}^{2 \csc \phi} \int_0^{2\pi} \rho^2 \sin \phi \, d\theta \, d\rho \, d\phi &= 2\pi \int_{\pi/6}^{\pi/3} \int_{\csc \phi}^{2 \csc \phi} \rho^2 \sin \phi \, d\rho \, d\phi = \frac{2\pi}{3} \int_{\pi/6}^{\pi/3} [\rho^3 \sin \phi]_{\csc \phi}^{2 \csc \phi} \, d\phi \\
&= \frac{14\pi}{3} \int_{\pi/6}^{\pi/3} \csc^2 \phi \, d\phi = \frac{28\pi}{3\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
29. \int_0^1 \int_0^\pi \int_0^{\pi/4} 12\rho \sin^3 \phi \, d\phi \, d\theta \, d\rho &= \int_0^1 \int_0^\pi \left(12\rho \left[-\frac{\sin^2 \phi \cos \phi}{3} \right]_0^{\pi/4} + 8\rho \int_0^{\pi/4} \sin \phi \, d\phi \right) d\theta \, d\rho \\
&= \int_0^1 \int_0^\pi \left(-\frac{2\rho}{\sqrt{2}} - 8\rho [\cos \phi]_0^{\pi/4} \right) d\theta \, d\rho = \int_0^1 \int_0^\pi \left(8\rho - \frac{10\rho}{\sqrt{2}} \right) d\theta \, d\rho = \pi \int_0^1 \left(8\rho - \frac{10\rho}{\sqrt{2}} \right) d\rho = \pi \left[4\rho^2 - \frac{5\rho^2}{\sqrt{2}} \right]_0^1 \\
&= \frac{(4\sqrt{2} - 5)\pi}{\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
30. \int_{\pi/6}^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_{\csc \phi}^2 5\rho^4 \sin^3 \phi \, d\rho \, d\theta \, d\phi &= \int_{\pi/6}^{\pi/2} \int_{-\pi/2}^{\pi/2} (32 - \csc^5 \phi) \sin^3 \phi \, d\theta \, d\phi = \int_{\pi/6}^{\pi/2} \int_{-\pi/2}^{\pi/2} (32 \sin^3 \phi - \csc^2 \phi) \, d\theta \, d\phi \\
&= \pi \int_{\pi/6}^{\pi/2} (32 \sin^3 \phi - \csc^2 \phi) \, d\phi = \pi \left[-\frac{32 \sin^2 \phi \cos \phi}{3} \right]_{\pi/6}^{\pi/2} + \frac{64\pi}{3} \int_{\pi/6}^{\pi/2} \sin \phi \, d\phi + \pi [\cot \phi]_{\pi/6}^{\pi/2} \\
&= \pi \left(\frac{32\sqrt{3}}{24} \right) - \frac{64\pi}{3} [\cos \phi]_{\pi/6}^{\pi/2} + \pi (\sqrt{3}) = \frac{\sqrt{3}}{3}\pi + \left(\frac{64\pi}{3} \right) \left(\frac{\sqrt{3}}{2} \right) = \frac{33\pi\sqrt{3}}{3} = 11\pi\sqrt{3}
\end{aligned}$$

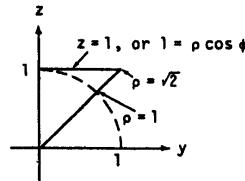
31. (a) $x^2 + y^2 = 1 \Rightarrow \rho^2 \sin^2 \phi = 1$, and $\rho \sin \phi = 1 \Rightarrow \rho = \csc \phi$; thus

$$\int_0^{2\pi} \int_0^{\pi/6} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta + \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^{\csc \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$(b) \int_0^{2\pi} \int_0^2 \int_{\pi/6}^{\sin^{-1}(1/\rho)} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta + \int_0^{2\pi} \int_0^2 \int_0^{\pi/6} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta$$

$$32. (a) \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$(b) \int_0^{2\pi} \int_0^1 \int_0^{\pi/4} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta + \int_0^{2\pi} \int_1^{\sqrt{2}} \int_{\cos^{-1}(1/\rho)}^{\pi/4} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta$$



$$33. V = \int_0^{2\pi} \int_0^{\pi/2} \int_{\cos \phi}^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/2} (8 - \cos^3 \phi) \sin \phi \, d\phi \, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \left[-8 \cos \phi + \frac{\cos^4 \phi}{4} \right]_0^{\pi/2} \, d\theta = \frac{1}{3} \int_0^{2\pi} \left(8 - \frac{1}{4} \right) \, d\theta = \left(\frac{31}{12} \right) (2\pi) = \frac{31\pi}{6}$$

$$34. V = \int_0^{2\pi} \int_0^{\pi/2} \int_1^{1+\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/2} (3 \cos \phi + 3 \cos^2 \phi + \cos^3 \phi) \sin \phi \, d\phi \, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \left[-\frac{3}{2} \cos^2 \phi - \cos^3 \phi - \frac{1}{4} \cos^4 \phi \right]_0^{\pi/2} \, d\theta = \frac{1}{3} \int_0^{2\pi} \left(\frac{3}{2} + 1 + \frac{1}{4} \right) \, d\theta = \frac{11}{12} \int_0^{2\pi} \, d\theta = \left(\frac{11}{12} \right) (2\pi) = \frac{11\pi}{6}$$

$$35. V = \int_0^{2\pi} \int_0^{\pi} \int_0^{1-\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi} (1 - \cos \phi)^3 \sin \phi \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \left[\frac{(1 - \cos \phi)^4}{4} \right]_0^{\pi} \, d\theta$$

$$= \frac{1}{12} (2)^4 \int_0^{2\pi} \, d\theta = \frac{4}{3} (2\pi) = \frac{8\pi}{3}$$

$$36. V = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{1-\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/2} (1 - \cos \phi)^3 \sin \phi \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \left[\frac{(1 - \cos \phi)^4}{4} \right]_0^{\pi/2} \, d\theta$$

$$= \frac{1}{12} \int_0^{2\pi} \, d\theta = \frac{1}{12} (2\pi) = \frac{\pi}{6}$$

$$37. V = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{2 \cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{8}{3} \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \cos^3 \phi \sin \phi \, d\phi \, d\theta = \frac{8}{3} \int_0^{2\pi} \left[-\frac{\cos^4 \phi}{4} \right]_{\pi/4}^{\pi/2} \, d\theta$$

$$= \left(\frac{8}{3} \right) \left(\frac{1}{16} \right) \int_0^{2\pi} \, d\theta = \frac{1}{6} (2\pi) = \frac{\pi}{3}$$

$$38. V = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{8}{3} \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \sin \phi \, d\phi \, d\theta = \frac{8}{3} \int_0^{2\pi} [-\cos \phi]_{\pi/3}^{\pi/2} \, d\theta = \frac{4}{3} \int_0^{2\pi} \, d\theta = \frac{8\pi}{3}$$

39. (a) $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

(b) $\int_0^{\pi/2} \int_0^2 \int_0^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta$

(c) $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} dz \, dy \, dx$

40. (a) $\int_0^{\pi/2} \int_0^{3/\sqrt{2}} \int_r^{\sqrt{9-r^2}} dz \, r \, dr \, d\theta$

(b) $\int_0^{\pi/2} \int_0^{\pi/4} \int_0^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

(c) $\int_0^{\pi/2} \int_0^{\pi/4} \int_0^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 9 \int_0^{\pi/2} \int_0^{\pi/4} \sin \phi \, d\phi \, d\theta = -9 \int_0^{\pi/2} \left(\frac{1}{\sqrt{2}} - 1 \right) d\theta = \frac{9\pi(2-\sqrt{2})}{4}$

41. (a) $V = \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec \phi}^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

(b) $V = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta$

(c) $V = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} dz \, dy \, dx$

(d) $V = \int_0^{2\pi} \int_0^{\sqrt{3}} \left[r(4-r^2)^{1/2} - r \right] dr \, d\theta = \int_0^{2\pi} \left[-\frac{(4-r^2)^{3/2}}{3} - \frac{r^2}{2} \right]_0^{\sqrt{3}} dr \, d\theta = \int_0^{2\pi} \left(-\frac{1}{3} - \frac{3}{2} + \frac{4^{3/2}}{3} \right) dr \, d\theta$
 $= \frac{5}{6} \int_0^{2\pi} dr \, d\theta = \frac{5\pi}{3}$

42. (a) $I_z = \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} r^2 \, dz \, r \, dr \, d\theta$

(b) $I_z = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (\rho^2 \sin^2 \phi)(\rho^2 \sin \phi) \, d\rho \, d\phi \, d\theta$, since $r^2 = x^2 + y^2 = \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta$
 $= \rho^2 \sin^2 \phi$

(c) $I_z = \int_0^{2\pi} \int_0^{\pi/2} \frac{1}{5} \sin^3 \phi \, d\phi \, d\theta = \frac{1}{5} \int_0^{2\pi} \left(\left[-\frac{\sin^2 \phi \cos \phi}{3} \right]_0^{\pi/2} + \frac{2}{3} \int_0^{\pi/2} \sin \phi \, d\phi \right) d\theta = \frac{2}{15} \int_0^{2\pi} [-\cos \phi]_0^{\pi/2} d\theta$
 $= \frac{2}{15}(2\pi) = \frac{4\pi}{15}$

43. $V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^4-1}^{4-4r^2} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 (5r - 4r^3 - r^5) dr \, d\theta = 4 \int_0^{\pi/2} \left(\frac{5}{2} - 1 - \frac{1}{6} \right) d\theta$

$$= 4 \int_0^{\pi/2} \frac{8}{6} d\theta = \frac{8\pi}{3}$$

$$44. V = 4 \int_0^{\pi/2} \int_0^1 \int_{-\sqrt{1-r^2}}^{1-r} dz r dr d\theta = 4 \int_0^{\pi/2} \int_0^1 (r - r^2 + r\sqrt{1-r^2}) dr d\theta = 4 \int_0^{\pi/2} \left[\frac{r^2}{2} - \frac{r^3}{3} - \frac{1}{3}(1-r^2)^{3/2} \right]_0^1 d\theta \\ = 4 \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3} \right) d\theta = 2 \int_0^{\pi/2} d\theta = 2\left(\frac{\pi}{2}\right) = \pi$$

$$45. V = \int_{3\pi/2}^{2\pi} \int_0^{3 \cos \theta} \int_0^{-r \sin \theta} dz r dr d\theta = \int_{3\pi/2}^{2\pi} \int_0^{3 \cos \theta} -r^2 \sin \theta dr d\theta = \int_{3\pi/2}^{2\pi} (-9 \cos^3 \theta)(\sin \theta) d\theta \\ = \left[\frac{9}{4} \cos^4 \theta \right]_{3\pi/2}^{2\pi} = \frac{9}{4} - 0 = \frac{9}{4}$$

$$46. V = 2 \int_{\pi/2}^{\pi} \int_0^{-3 \cos \theta} \int_0^r dz r dr d\theta = 2 \int_{\pi/2}^{\pi} \int_0^{-3 \cos \theta} r^2 dr d\theta = \frac{2}{3} \int_{\pi/2}^{\pi} -27 \cos^3 \theta d\theta \\ = -18 \left(\left[\frac{\cos^2 \theta \sin \theta}{3} \right]_{\pi/2}^{\pi} + \frac{2}{3} \int_{\pi/2}^{\pi} \cos \theta d\theta \right) = -12[\sin \theta]_{\pi/2}^{\pi} = 12$$

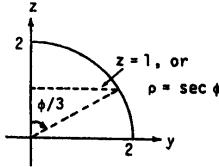
$$47. V = \int_0^{\pi/2} \int_0^{\sin \theta} \int_0^{\sqrt{1-r^2}} dz r dr d\theta = \int_0^{\pi/2} \int_0^{\sin \theta} r \sqrt{1-r^2} dr d\theta = \int_0^{\pi/2} \left[-\frac{1}{3}(1-r^2)^{3/2} \right]_0^{\sin \theta} d\theta \\ = -\frac{1}{3} \int_0^{\pi/2} \left[(1-\sin^2 \theta)^{3/2} - 1 \right] d\theta = -\frac{1}{3} \int_0^{\pi/2} (\cos^3 \theta - 1) d\theta = -\frac{1}{3} \left(\left[\frac{\cos^2 \theta \sin \theta}{3} \right]_0^{\pi/2} + \frac{2}{3} \int_0^{\pi/2} \cos \theta d\theta \right) + \left[\frac{\theta}{3} \right]_0^{\pi/2} \\ = -\frac{2}{9}[\sin \theta]_0^{\pi/2} + \frac{\pi}{6} = \frac{-4+3\pi}{18}$$

$$48. V = \int_0^{\pi/2} \int_0^{\cos \theta} \int_0^{3\sqrt{1-r^2}} dz r dr d\theta = \int_0^{\pi/2} \int_0^{\cos \theta} 3r\sqrt{1-r^2} dr d\theta = \int_0^{\pi/2} \left[-(1-r^2)^{3/2} \right]_0^{\cos \theta} d\theta \\ = \int_0^{\pi/2} \left[-(1-\cos^2 \theta)^{3/2} + 1 \right] d\theta = \int_0^{\pi/2} (1-\sin^3 \theta) d\theta = \left[\theta + \frac{\sin^2 \theta \cos \theta}{3} \right]_0^{\pi/2} - \frac{2}{3} \int_0^{\pi/2} \sin \theta d\theta \\ = \frac{\pi}{2} + \frac{2}{3}[\cos \theta]_0^{\pi/2} = \frac{\pi}{2} - \frac{2}{3} = \frac{3\pi-4}{6}$$

$$49. V = \int_0^{2\pi} \int_{\pi/3}^{2\pi/3} \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_{\pi/3}^{2\pi/3} \frac{a^3}{3} \sin \phi \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{2\pi} [-\cos \phi]_{\pi/3}^{2\pi/3} \, d\theta = \frac{a^3}{3} \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}\right) \, d\theta = \frac{2\pi a^3}{3}$$

$$50. V = \int_0^{\pi/6} \int_0^{\pi/2} \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{\pi/6} \int_0^{\pi/2} \sin \phi \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{\pi/6} \int_0^{\pi/2} d\theta = \frac{a^3 \pi}{18}$$

$$\begin{aligned} 51. V &= \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec \phi}^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/3} (8 \sin \phi - \tan \phi \sec^2 \phi) \, d\phi \, d\theta \\ &= \frac{1}{3} \int_0^{2\pi} \left[-8 \cos \phi - \frac{1}{2} \tan^2 \phi \right]_0^{\pi/3} \, d\theta \end{aligned}$$



$$= \frac{1}{3} \int_0^{2\pi} \left[-4 - \frac{1}{2}(3) + 8 \right] \, d\theta = \frac{1}{3} \int_0^{2\pi} \frac{5}{2} \, d\theta = \frac{5}{6}(2\pi) = \frac{5\pi}{3}$$

$$\begin{aligned} 52. V &= 4 \int_0^{\pi/2} \int_0^{\pi/4} \int_{\sec \phi}^{2 \sec \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{4}{3} \int_0^{\pi/2} \int_0^{\pi/4} (8 \sec^3 \phi - \sec^3 \phi) \sin \phi \, d\phi \, d\theta \\ &= \frac{28}{3} \int_0^{\pi/2} \int_0^{\pi/4} \sec^3 \phi \sin \phi \, d\phi \, d\theta = \frac{28}{3} \int_0^{\pi/2} \int_0^{\pi/4} \tan \phi \sec^2 \phi \, d\phi \, d\theta = \frac{28}{3} \int_0^{\pi/2} \left[\frac{1}{2} \tan^2 \phi \right]_0^{\pi/4} \, d\theta \\ &= \frac{14}{3} \int_0^{\pi/2} \, d\theta = \frac{7\pi}{3} \end{aligned}$$

$$53. V = 4 \int_0^{\pi/2} \int_0^1 \int_0^{r^2} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 r^3 \, dr \, d\theta = \int_0^{\pi/2} \, d\theta = \frac{\pi}{2}$$

$$54. V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{r^2+1} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 r \, dr \, d\theta = 2 \int_0^{\pi/2} \, d\theta = \pi$$

$$55. V = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} \int_0^r dz \, r \, dr \, d\theta = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} r^2 \, dr \, d\theta = 8 \left(\frac{2\sqrt{2}-1}{3} \right) \int_0^{\pi/2} \, d\theta = \frac{4\pi(2\sqrt{2}-1)}{3}$$

$$56. V = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} \int_0^{\sqrt{2-r^2}} dz r dr d\theta = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} r \sqrt{2-r^2} dr d\theta = 8 \int_0^{\pi/2} \left[-\frac{1}{3}(2-r^2)^{3/2} \right]_1^{\sqrt{2}} d\theta \\ = \frac{8}{3} \int_0^{\pi/2} d\theta = \frac{4\pi}{3}$$

$$57. V = \int_0^{2\pi} \int_0^2 \int_0^{4-r \sin \theta} dz r dr d\theta = \int_0^{2\pi} \int_0^2 (4r - r^2 \sin \theta) dr d\theta = 8 \int_0^{2\pi} \left(1 - \frac{\sin \theta}{3} \right) d\theta = 16\pi$$

$$58. V = \int_0^{2\pi} \int_0^2 \int_0^{4-r \cos \theta - r \sin \theta} dz r dr d\theta = \int_0^{2\pi} \int_0^2 [4r - r^2(\cos \theta + \sin \theta)] dr d\theta = \frac{8}{3} \int_0^{2\pi} (3 - \cos \theta - \sin \theta) d\theta = 16\pi$$

59. The paraboloids intersect when $4x^2 + 4y^2 = 5 - x^2 - y^2 \Rightarrow x^2 + y^2 = 1$ and $z = 4$

$$\Rightarrow V = 4 \int_0^{\pi/2} \int_0^1 \int_{\frac{5-r^2}{4r^2}}^{5-r^2} dz r dr d\theta = 4 \int_0^{\pi/2} \int_0^1 (5r - 5r^3) dr d\theta = 20 \int_0^{\pi/2} \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 d\theta = 5 \int_0^{\pi/2} d\theta = \frac{5\pi}{2}$$

60. The paraboloid intersects the xy-plane when $9 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 9 \Rightarrow$

$$V = 4 \int_0^{\pi/2} \int_1^3 \int_0^{9-r^2} dz r dr d\theta = 4 \int_0^{\pi/2} \int_1^3 (9r - r^3) dr d\theta = 4 \int_0^{\pi/2} \left[\frac{9r^2}{2} - \frac{r^4}{4} \right]_1^3 d\theta = 4 \int_0^{\pi/2} \left(\frac{81}{4} - \frac{17}{4} \right) d\theta \\ = 64 \int_0^{\pi/2} d\theta = 32\pi$$

$$61. V = 8 \int_0^{\pi/2} \int_0^1 \int_0^{\sqrt{4-r^2}} dz r dr d\theta = 8 \int_0^{\pi/2} \int_0^1 r(4-r^2)^{1/2} dr d\theta = 8 \int_0^{\pi/2} \left[-\frac{1}{3}(4-r^2)^{3/2} \right]_0^1 d\theta \\ = -\frac{8}{3} \int_0^{\pi/2} (3^{3/2} - 8) d\theta = \frac{4\pi(8-3\sqrt{3})}{3}$$

62. The sphere and paraboloid intersect when $x^2 + y^2 + z^2 = 2$ and $z = x^2 + y^2 \Rightarrow z^2 + z - 2 = 0$

$\Rightarrow (z+2)(z-1) = 0 \Rightarrow z = 1$ or $z = -2 \Rightarrow z = 1$ since $z \geq 0$. Thus, $x^2 + y^2 = 1$ and the volume is

given by the triple integral $V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} dz r dr d\theta = 4 \int_0^{\pi/2} \int_0^1 [r(2-r^2)^{1/2} - r^3] dr d\theta$

$$= 4 \int_0^{\pi/2} \left[-\frac{1}{3}(2-r^2)^{3/2} - \frac{r^4}{4} \right]_0^1 d\theta = 4 \int_0^{\pi/2} \left(\frac{2\sqrt{2}}{3} - \frac{7}{12} \right) d\theta = \frac{\pi(8\sqrt{2}-7)}{6}$$

$$63. \text{ average} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \int_{-1}^1 r^2 dz dr d\theta = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 2r^2 dr d\theta = \frac{1}{3\pi} \int_0^{2\pi} d\theta = \frac{2}{3}$$

$$64. \text{ average} = \left(\frac{1}{\frac{4\pi}{3}} \right) \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^2 dz dr d\theta = \frac{3}{4\pi} \int_0^{2\pi} \int_0^1 2r^2 \sqrt{1-r^2} dr d\theta \\ = \frac{3}{2\pi} \int_0^{2\pi} \left[\frac{1}{8} \sin^{-1} r - \frac{1}{8} r \sqrt{1-r^2} (1-r^2) \right]_0^1 d\theta = \frac{3}{16\pi} \int_0^{2\pi} \left(\frac{\pi}{2} + 0 \right) d\theta = \frac{3}{32} \int_0^{2\pi} d\theta = \left(\frac{3}{32} \right) (2\pi) = \frac{3\pi}{16}$$

$$65. \text{ average} = \left(\frac{1}{\frac{4\pi}{3}} \right) \int_0^{2\pi} \int_0^\pi \int_0^1 \rho^3 \sin \phi d\rho d\phi d\theta = \frac{3}{16\pi} \int_0^{2\pi} \int_0^\pi \sin \phi d\phi d\theta = \frac{3}{8\pi} \int_0^{2\pi} d\theta = \frac{3}{4}$$

$$66. \text{ average} = \left(\frac{1}{\frac{2\pi}{3}} \right) \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^3 \cos \phi \sin \phi d\rho d\phi d\theta = \frac{3}{8\pi} \int_0^{2\pi} \int_0^{\pi/2} \cos \phi \sin \phi d\phi d\theta = \frac{3}{8\pi} \int_0^{2\pi} \left[\frac{\sin^2 \phi}{2} \right]_0^{\pi/2} d\theta \\ = \frac{3}{16\pi} \int_0^{2\pi} d\theta = \left(\frac{3}{16\pi} \right) (2\pi) = \frac{3}{8}$$

$$67. M = 4 \int_0^{\pi/2} \int_0^1 \int_0^r dz r dr d\theta = 4 \int_0^{\pi/2} \int_0^1 r^2 dr d\theta = \frac{4}{3} \int_0^{\pi/2} d\theta = \frac{2\pi}{3}; M_{xy} = \int_0^{2\pi} \int_0^1 \int_0^r z dz r dr d\theta \\ = \frac{1}{2} \int_0^{2\pi} \int_0^1 r^3 dr d\theta = \frac{1}{8} \int_0^{2\pi} d\theta = \frac{\pi}{4} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{\pi}{4} \right) \left(\frac{3}{2\pi} \right) = \frac{3}{8}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}$$

$$68. M = \int_0^{\pi/2} \int_0^2 \int_0^r dz r dr d\theta = \int_0^{\pi/2} \int_0^2 r^2 dr d\theta = \frac{8}{3} \int_0^{\pi/2} d\theta = \frac{4\pi}{3}; M_{yz} = \int_0^{\pi/2} \int_0^2 \int_0^r x dz r dr d\theta \\ = \int_0^{\pi/2} \int_0^2 r^3 \cos \theta dr d\theta = 4 \int_0^{\pi/2} \cos \theta d\theta = 4; M_{xz} = \int_0^{\pi/2} \int_0^2 \int_0^r y dz r dr d\theta = \int_0^{\pi/2} \int_0^2 r^3 \sin \theta dr d\theta \\ = 4 \int_0^{\pi/2} \sin \theta d\theta = 4; M_{xy} = \int_0^{\pi/2} \int_0^2 \int_0^r z dz r dr d\theta = \frac{1}{2} \int_0^{\pi/2} \int_0^2 r^3 dr d\theta = 2 \int_0^{\pi/2} d\theta = \pi \Rightarrow \bar{x} = \frac{M_{yz}}{M} = \frac{3}{\pi}, \\ \bar{y} = \frac{M_{xz}}{M} = \frac{3}{\pi}, \text{ and } \bar{z} = \frac{M_{xy}}{M} = \frac{3}{4}$$

$$69. M = \frac{8\pi}{3}; M_{xy} = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 z \rho^2 \sin \phi d\rho d\phi d\theta = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 \rho^3 \cos \phi \sin \phi d\rho d\phi d\theta = 4 \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \cos \phi \sin \phi d\phi d\theta$$

$$= 4 \int_0^{2\pi} \left[\frac{\sin^2 \phi}{2} \right]_{\pi/3}^{\pi/2} d\theta = 4 \int_0^{2\pi} \left(\frac{1}{2} - \frac{3}{8} \right) d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \pi \Rightarrow \bar{z} = \frac{M_{xy}}{M} = (\pi) \left(\frac{3}{8\pi} \right) = \frac{3}{8}, \text{ and } \bar{x} = \bar{y} = 0,$$

by symmetry

$$70. M = \int_0^{2\pi} \int_0^{\pi/4} \int_0^a \rho^2 \sin \phi d\rho d\phi d\theta = \frac{a^3}{3} \int_0^{2\pi} \int_0^{\pi/4} \sin \phi d\phi d\theta = \frac{a^3}{3} \int_0^{2\pi} \frac{2-\sqrt{2}}{2} d\theta = \frac{\pi a^3 (2-\sqrt{2})}{3};$$

$$M_{xy} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^a \rho^3 \sin \phi \cos \phi d\rho d\phi d\theta = \frac{a^3}{4} \int_0^{2\pi} \int_0^{\pi/4} \sin \phi \cos \phi d\phi d\theta = \frac{a^4}{16} \int_0^{2\pi} d\theta = \frac{\pi a^4}{8}$$

$$\Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{\pi a^4}{8} \right) \left[\frac{3}{\pi a^3 (2-\sqrt{2})} \right] = \left(\frac{3a}{8} \right) \left(\frac{2+\sqrt{2}}{2} \right) = \frac{3(2+\sqrt{2})a}{16}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}$$

$$71. M = \int_0^{2\pi} \int_0^4 \int_0^{\sqrt{r}} dz r dr d\theta = \int_0^{2\pi} \int_0^4 r^{3/2} dr d\theta = \frac{64}{5} \int_0^{2\pi} d\theta = \frac{128\pi}{5}; M_{xy} = \int_0^{2\pi} \int_0^4 \int_0^{\sqrt{r}} z dz r dr d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^4 r^2 dr d\theta = \frac{32}{3} \int_0^{2\pi} d\theta = \frac{64\pi}{3} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \frac{5}{6}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}$$

$$72. M = \int_{-\pi/3}^{\pi/3} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} dz r dr d\theta = \int_{-\pi/3}^{\pi/3} \int_0^1 2r\sqrt{1-r^2} dr d\theta = \int_{-\pi/3}^{\pi/3} \left[-\frac{2}{3}(1-r^2)^{3/2} \right]_0^1 d\theta$$

$$= \frac{2}{3} \int_{-\pi/3}^{\pi/3} d\theta = \left(\frac{2}{3} \right) \left(\frac{2\pi}{3} \right) = \frac{4\pi}{9}; M_{yz} = \int_{-\pi/3}^{\pi/3} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^2 \cos \theta dz dr d\theta = 2 \int_{-\pi/3}^{\pi/3} \int_0^1 r^2 \sqrt{1-r^2} \cos \theta dr d\theta$$

$$= 2 \int_{-\pi/3}^{\pi/3} \left[\frac{1}{8} \sin^{-1} r - \frac{1}{8} r \sqrt{1-r^2} (1-2r^2) \right]_0^1 \cos \theta d\theta = \frac{\pi}{8} \int_{-\pi/3}^{\pi/3} \cos \theta d\theta = \frac{\pi}{8} [\sin \theta]_{-\pi/3}^{\pi/3} = \left(\frac{\pi}{8} \right) \left(2 \cdot \frac{\sqrt{3}}{2} \right) = \frac{\pi\sqrt{3}}{8}$$

$$\Rightarrow \bar{x} = \frac{M_{yz}}{M} = \frac{9\sqrt{3}}{32}, \text{ and } \bar{y} = \bar{z} = 0, \text{ by symmetry}$$

$$73. I_z = \int_0^{2\pi} \int_1^2 \int_0^4 (x^2 + y^2) dz r dr d\theta = 4 \int_0^{2\pi} \int_1^2 r^3 dr d\theta = \int_0^{2\pi} 15 d\theta = 30\pi; M = \int_0^{2\pi} \int_1^2 \int_0^4 dz r dr d\theta$$

$$= \int_0^{2\pi} \int_1^2 4r dr d\theta = \int_0^{2\pi} 6 d\theta = 12\pi \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{5}{2}}$$

$$74. (a) I_z = \int_0^{2\pi} \int_0^1 \int_{-1}^1 r^3 dz dr d\theta = 2 \int_0^{2\pi} \int_0^1 r^3 dr d\theta = \frac{1}{2} \int_0^{2\pi} r^4 d\theta = \pi$$

$$(b) I_x = \int_0^{2\pi} \int_0^1 \int_{-1}^1 (r^2 \sin^2 \theta + z^2) dz r dr d\theta = 2 \int_0^{2\pi} \int_0^1 \left(2r^3 \sin^2 \theta + \frac{2r}{3} \right) dr d\theta = \int_0^{2\pi} \left(\frac{\sin^2 \theta}{2} + \frac{1}{3} \right) d\theta$$

$$= \left[\frac{\theta}{4} - \frac{\sin 2\theta}{8} + \frac{\theta}{3} \right]_0^{2\pi} = \frac{\pi}{2} + \frac{2\pi}{3} = \frac{7\pi}{6}$$

75. We orient the cone with its vertex at the origin and axis along the z-axis $\Rightarrow \phi = \frac{\pi}{4}$. We use the x-axis

$$\text{which is through the vertex and parallel to the base of the cone} \Rightarrow I_x = \int_0^{2\pi} \int_0^1 \int_r^1 (r^2 \sin^2 \theta + z^2) dz r dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left(r^3 \sin^2 \theta - r^4 \sin^2 \theta + \frac{r}{3} - \frac{r^4}{4} \right) dr d\theta = \int_0^{2\pi} \left(\frac{\sin^2 \theta}{20} + \frac{1}{10} \right) d\theta = \left[\frac{\theta}{40} - \frac{\sin 2\theta}{80} + \frac{\theta}{10} \right]_0^{2\pi} = \frac{\pi}{20} + \frac{\pi}{5} = \frac{\pi}{4}$$

$$76. I_z = \int_0^{2\pi} \int_0^a \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^3 dz dr d\theta = \int_0^{2\pi} \int_0^a 2r^3 \sqrt{a^2-r^2} dr d\theta = 2 \int_0^{2\pi} \left[\left(-\frac{r^2}{5} - \frac{2a^2}{15} \right) (a^2 - r^2)^{3/2} \right]_0^a d\theta$$

$$= 2 \int_0^{2\pi} \frac{2}{15} a^5 d\theta = \frac{8\pi a^5}{15}$$

$$77. I_z = \int_0^{2\pi} \int_0^a \int_{(\frac{h}{a})r}^h (x^2 + y^2) dz r dr d\theta = \int_0^{2\pi} \int_0^a \int_{\frac{hr}{a}}^h r^3 dz dr d\theta = \int_0^{2\pi} \int_0^a \left(hr^3 - \frac{hr^4}{a} \right) dr d\theta$$

$$= \int_0^{2\pi} h \left[\frac{r^4}{4} - \frac{r^5}{5a} \right]_0^a d\theta = \int_0^{2\pi} h \left(\frac{a^4}{4} - \frac{a^5}{5a} \right) d\theta = \frac{ha^4}{20} \int_0^{2\pi} d\theta = \frac{\pi ha^4}{10}$$

$$78. (a) M = \int_0^{2\pi} \int_0^1 \int_0^r z dz r dr d\theta = \int_0^{2\pi} \int_0^1 \frac{1}{2} r^5 dr d\theta = \frac{1}{12} \int_0^{2\pi} d\theta = \frac{\pi}{6}; M_{xy} = \int_0^{2\pi} \int_0^1 \int_0^r z^2 dz r dr d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \int_0^1 r^7 dr d\theta = \frac{1}{24} \int_0^{2\pi} d\theta = \frac{\pi}{12} \Rightarrow \bar{z} = \frac{1}{2}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry;}$$

$$I_z = \int_0^{2\pi} \int_0^1 \int_0^r zr^3 dz dr d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^1 r^7 dr d\theta = \frac{1}{16} \int_0^{2\pi} d\theta = \frac{\pi}{8} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \frac{\sqrt{3}}{2}$$

$$(b) M = \int_0^{2\pi} \int_0^1 \int_0^r r^2 dz dr d\theta = \int_0^{2\pi} \int_0^1 r^4 dr d\theta = \frac{1}{5} \int_0^{2\pi} d\theta = \frac{2\pi}{5}; M_{xy} = \int_0^{2\pi} \int_0^1 \int_0^r zr^2 dz dr d\theta$$

$$\begin{aligned}
&= \frac{1}{2} \int_0^{2\pi} \int_0^1 r^6 dr d\theta = \frac{1}{14} \int_0^{2\pi} d\theta = \frac{\pi}{7} \Rightarrow \bar{z} = \frac{5}{14}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry; } I_z = \int_0^{2\pi} \int_0^1 \int_0^{r^2} r^4 dz dr d\theta \\
&= \int_0^{2\pi} \int_0^1 r^6 dr d\theta = \frac{1}{7} \int_0^{2\pi} d\theta = \frac{2\pi}{7} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{5}{7}}
\end{aligned}$$

$$\begin{aligned}
79. \text{ (a)} \quad M &= \int_0^{2\pi} \int_0^1 \int_r^1 z dz r dr d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^1 (r - r^3) dr d\theta = \frac{1}{8} \int_0^{2\pi} d\theta = \frac{\pi}{4}; \quad M_{xy} = \int_0^{2\pi} \int_0^1 \int_r^1 z^2 dz r dr d\theta \\
&= \frac{1}{3} \int_0^{2\pi} \int_0^1 (r - r^4) dr d\theta = \frac{1}{10} \int_0^{2\pi} d\theta = \frac{\pi}{5} \Rightarrow \bar{z} = \frac{4}{5}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry; } I_z = \int_0^{2\pi} \int_0^1 \int_r^1 zr^3 dz dr d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \int_0^1 (r^3 - r^5) dr d\theta = \frac{1}{24} \int_0^{2\pi} d\theta = \frac{\pi}{12} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{1}{3}} \\
\text{(b)} \quad M &= \int_0^{2\pi} \int_0^1 \int_r^1 z^2 dz r dr d\theta = \frac{\pi}{5} \text{ from part (a); } M_{xy} = \int_0^{2\pi} \int_0^1 \int_r^1 z^3 dz r dr d\theta = \frac{1}{4} \int_0^{2\pi} \int_0^1 (r - r^5) dr d\theta \\
&= \frac{1}{12} \int_0^{2\pi} d\theta = \frac{\pi}{6} \Rightarrow \bar{z} = \frac{5}{6}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry; } I_z = \int_0^{2\pi} \int_0^1 \int_r^1 z^2 r^3 dz dr d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^1 (r^3 - r^6) dr d\theta \\
&= \frac{1}{28} \int_0^{2\pi} d\theta = \frac{\pi}{14} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{5}{14}}
\end{aligned}$$

$$\begin{aligned}
80. \text{ (a)} \quad M &= \int_0^{2\pi} \int_0^\pi \int_0^a \rho^4 \sin \phi d\rho d\phi d\theta = \frac{a^5}{5} \int_0^{2\pi} \int_0^\pi \sin \phi d\phi d\theta = \frac{2a^5}{5} \int_0^{2\pi} d\theta = \frac{4\pi a^5}{5}; \\
I_z &= \int_0^{2\pi} \int_0^\pi \int_0^a \rho^6 \sin^3 \phi d\rho d\phi d\theta = \frac{a^7}{7} \int_0^{2\pi} \int_0^\pi (1 - \cos^2 \phi) \sin \phi d\phi d\theta = \frac{a^7}{7} \int_0^{2\pi} \left[-\cos \phi + \frac{\cos^3 \phi}{3} \right]_0^\pi d\theta \\
&= \frac{4a^7}{21} \int_0^{2\pi} d\theta = \frac{8a^7 \pi}{21} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{10}{21}} a \\
\text{(b)} \quad M &= \int_0^{2\pi} \int_0^\pi \int_0^a \rho^3 \sin^2 \phi d\rho d\phi d\theta = \frac{a^4}{4} \int_0^{2\pi} \int_0^\pi \frac{(1 - \cos 2\phi)}{2} d\phi d\theta = \frac{\pi a^4}{8} \int_0^{2\pi} d\theta = \frac{\pi^2 a^4}{4};
\end{aligned}$$

$$\begin{aligned}
&= \frac{a^6}{6} \int_0^{2\pi} \left(\left[-\frac{\sin^3 \phi \cos \phi}{4} \right]_0^\pi + \frac{3}{4} \int_0^\pi \sin^2 \phi \, d\phi \right) d\theta = \frac{a^6}{8} \int_0^{2\pi} \left[\frac{\phi}{2} - \frac{\sin 2\phi}{4} \right]_0^\pi d\theta = \frac{\pi a^6}{16} \int_0^{2\pi} d\theta \\
&= \frac{a^6 \pi^2}{8} \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \frac{a}{\sqrt{2}}
\end{aligned}$$

81. $M = \int_0^{2\pi} \int_0^a \int_0^{\frac{h}{a} \sqrt{a^2 - r^2}} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^a \frac{h}{a} r \sqrt{a^2 - r^2} \, dr \, d\theta = \frac{h}{a} \int_0^{2\pi} \left[-\frac{1}{3}(a^2 - r^2)^{3/2} \right]_0^a d\theta$

$$\begin{aligned}
&= \frac{h}{a} \int_0^{2\pi} \frac{a^3}{3} d\theta = \frac{2ha^2\pi}{3}; M_{xy} = \int_0^{2\pi} \int_0^a \int_0^{\frac{h}{a} \sqrt{a^2 - r^2}} z \, dz \, r \, dr \, d\theta = \frac{h^2}{2a^2} \int_0^{2\pi} (a^2r - r^3) \, dr \, d\theta \\
&= \frac{h^2}{2a^2} \int_0^{2\pi} \left(\frac{a^4}{2} - \frac{a^4}{4} \right) d\theta = \frac{a^2 h^2 \pi}{4} \Rightarrow \bar{z} = \left(\frac{\pi a^2 h^2}{4} \right) \left(\frac{3}{2ha^2\pi} \right) = \frac{3}{8}h, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry}
\end{aligned}$$

82. Let the base radius of the cone be a and the height h , and place the cone's axis of symmetry along the z -axis

with the vertex at the origin. Then $M = \frac{\pi r^2 h}{3}$ and $M_{xy} = \int_0^{2\pi} \int_0^a \int_0^h z \, dz \, r \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^a \left(h^2 r - \frac{h^2}{a^2} r^3 \right) dr \, d\theta$

$$\begin{aligned}
&= \frac{h^2}{2} \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4a^2} \right]_0^a d\theta = \frac{h^2}{2} \int_0^{2\pi} \left(\frac{a^2}{2} - \frac{a^2}{4} \right) d\theta = \frac{h^2 a^2}{8} \int_0^{2\pi} d\theta = \frac{h^2 a^2 \pi}{4} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{h^2 a^2 \pi}{4} \right) \left(\frac{3}{\pi a^2 h} \right) = \frac{3}{4}h, \text{ and} \\
&\bar{x} = \bar{y} = 0, \text{ by symmetry} \Rightarrow \text{the centroid is one fourth of the way from the base to the vertex}
\end{aligned}$$

83. $M = \int_0^{2\pi} \int_0^a \int_0^h (z+1) \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^a \left(\frac{h^2}{2} + h \right) r \, dr \, d\theta = \frac{a^2(h^2 + 2h)}{4} \int_0^{2\pi} d\theta = \frac{\pi a^2(h^2 + 2h)}{2};$

$$\begin{aligned}
M_{xy} &= \int_0^{2\pi} \int_0^a \int_0^h (z^2 + z) \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^a \left(\frac{h^3}{3} + \frac{h^2}{2} \right) r \, dr \, d\theta = \frac{a^2(2h^3 + 3h^2)}{12} \int_0^{2\pi} d\theta = \frac{\pi a^2(2h^3 + 3h^2)}{6} \\
\Rightarrow \bar{z} &= \left[\frac{\pi a^2(2h^3 + 3h^2)}{6} \right] \left[\frac{2}{\pi a^2(h^2 + 2h)} \right] = \frac{2h^2 + 3h}{3h + 6}, \text{ and } \bar{x} = \bar{y} = 0, \text{ by symmetry;} \\
I_z &= \int_0^{2\pi} \int_0^a \int_0^h (z+1)r^3 \, dz \, dr \, d\theta = \left(\frac{h^2 + 2h}{2} \right) \int_0^{2\pi} \int_0^a r^3 \, dr \, d\theta = \left(\frac{h^2 + 2h}{2} \right) \left(\frac{a^4}{4} \right) \int_0^{2\pi} d\theta = \frac{\pi a^4(h^2 + 2h)}{4}; \\
R_z &= \sqrt{\frac{I_z}{M}} = \sqrt{\frac{\pi a^4(h^2 + 2h)}{4} \cdot \frac{2}{\pi a^2(h^2 + 2h)}} = \frac{a}{\sqrt{2}}
\end{aligned}$$

84. The mass of the planet's atmosphere to an altitude h above the surface of the planet is the triple integral

$$\begin{aligned} M(h) &= \int_0^{2\pi} \int_0^\pi \int_R^h \mu_0 e^{-c(\rho-R)} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_R^h \int_0^{2\pi} \int_0^\pi \mu_0 e^{-c(\rho-R)} \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho \\ &= \int_R^h \int_0^{2\pi} \left[\mu_0 e^{-c(\rho-R)} \rho^2 (-\cos \phi) \right]_0^\pi \, d\theta \, d\rho = 2 \int_R^h \int_0^{2\pi} \mu_0 e^{cR} e^{-c\rho} \rho^2 \, d\theta \, d\rho = 4\pi \mu_0 e^{cR} \int_R^h e^{-c\rho} \rho^2 \, d\rho \\ &= 4\pi \mu_0 e^{cR} \left[-\frac{\rho^2 e^{-c\rho}}{c} - \frac{2\rho e^{-c\rho}}{c^2} - \frac{2e^{-c\rho}}{c^3} \right]_R^h \quad (\text{by parts}) \\ &= 4\pi \mu_0 e^{cR} \left(-\frac{h^2 e^{-ch}}{c} - \frac{2he^{-ch}}{c^2} - \frac{2e^{-ch}}{c^3} + \frac{R^2 e^{-cR}}{c} + \frac{2Re^{-cR}}{c^2} + \frac{2e^{-cR}}{c^3} \right). \end{aligned}$$

The mass of the planet's atmosphere is therefore $M = \lim_{h \rightarrow \infty} M(h) = 4\pi \mu_0 \left(\frac{R^2}{c} + \frac{2R}{c^2} + \frac{2}{c^3} \right)$.

85. The density distribution function is linear so it has the form $\delta(\rho) = k\rho + C$, where ρ is the distance from the center of the planet. Now, $\delta(R) = 0 \Rightarrow kR + C = 0$, and $\delta(\rho) = k\rho - kR$. It remains to determine the constant

$$\begin{aligned} k: M &= \int_0^{2\pi} \int_0^\pi \int_0^R (k\rho - kR) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^\pi \left[k \frac{\rho^4}{4} - kR \frac{\rho^3}{3} \right]_0^R \sin \phi \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^\pi k \left(\frac{R^4}{4} - \frac{R^4}{3} \right) \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} -\frac{k}{12} R^4 [-\cos \phi]_0^\pi \, d\theta = \int_0^{2\pi} -\frac{k}{6} R^4 \, d\theta = -\frac{k\pi R^4}{3} \Rightarrow k = -\frac{3M}{\pi R^4} \\ &\Rightarrow \delta(\rho) = -\frac{3M}{\pi R^4} \rho + \frac{3M}{\pi R^4} R. \text{ At the center of the planet } \rho = 0 \Rightarrow \delta(0) = \left(\frac{3M}{\pi R^4} \right) R = \frac{3M}{\pi R^3}. \end{aligned}$$

86. $x^2 + y^2 = a^2 \Rightarrow (\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2 = a^2 \Rightarrow (\rho^2 \sin^2 \phi)(\cos^2 \theta + \sin^2 \theta) = a^2 \Rightarrow \rho^2 \sin^2 \phi = a^2$
 $\Rightarrow \rho \sin \phi = a \text{ or } \rho \sin \phi = -a \Rightarrow \rho \sin \phi = a \text{ or } \rho = a \csc \phi, \text{ since } 0 \leq \phi \leq \pi \text{ and } \rho \geq 0$

87. (a) A plane perpendicular to the x -axis has the form $x = a$ in rectangular coordinates $\Rightarrow r \cos \theta = a$

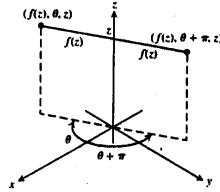
$$\Rightarrow r = \frac{a}{\cos \theta} \Rightarrow r = a \sec \theta, \text{ in cylindrical coordinates.}$$

- (b) A plane perpendicular to the y -axis has the form $y = b$ in rectangular coordinates $\Rightarrow r \sin \theta = b$

$$\Rightarrow r = \frac{b}{\sin \theta} \Rightarrow r = b \csc \theta, \text{ in cylindrical coordinates.}$$

88. $ax + by = c \Rightarrow a(r \cos \theta) + b(r \sin \theta) = c \Rightarrow r(a \cos \theta + b \sin \theta) = c \Rightarrow r = \frac{c}{a \cos \theta + b \sin \theta}$

89. The equation $r = f(z)$ implies that the point $(r, \theta, z) = (f(z), \theta, z)$ will lie on the surface for all θ . In particular $(f(z), \theta + \pi, z)$ lies on the surface whenever $(f(z), \theta, z)$ does \Rightarrow the surface is symmetric with respect to the z -axis.



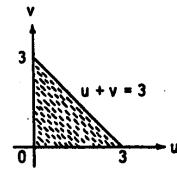
90. The equation $\rho = f(\phi)$ implies that the point $(\rho, \phi, \theta) = (f(\phi), \phi, \theta)$ lies on the surface for all θ . In particular, if $(f(\phi), \phi, \theta)$ lies on the surface, then $(f(\phi), \phi, \theta + \pi)$ lies on the surface, so the surface is symmetric with respect to the z -axis.

12.7 SUBSTITUTIONS IN MULTIPLE INTEGRALS

1. (a) $x - y = u$ and $2x + y = v \Rightarrow 3x = u + v$ and $y = x - u \Rightarrow x = \frac{1}{3}(u + v)$ and $y = \frac{1}{3}(-2u + v)$;

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -2 & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$$

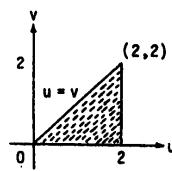
- (b) The line segment $y = x$ from $(0,0)$ to $(1,1)$ is $x - y = 0$
 $\Rightarrow u = 0$; the line segment $y = -2x$ from $(0,0)$ to $(1,-2)$
 $\text{is } 2x + y = 0 \Rightarrow v = 0$; the line segment $x = 1$ from
 $(1,1)$ to $(1,-2)$ is $(x-y) + (2x+y) = 3 \Rightarrow u+v = 3$.
The transformed region is sketched at the right.



2. (a) $x + 2y = u$ and $x - y = v \Rightarrow 3y = u - v$ and $x = v + y \Rightarrow y = \frac{1}{3}(u - v)$ and $x = \frac{1}{3}(u + 2v)$;

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$$

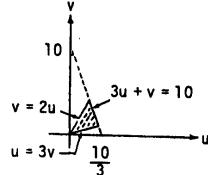
- (b) The triangular region in the xy -plane has vertices $(0,0)$, $(2,0)$, and $(\frac{2}{3}, \frac{2}{3})$. The line segment $y = x$ from $(0,0)$ to $(\frac{2}{3}, \frac{2}{3})$ is $x - y = 0 \Rightarrow v = 0$; the line segment $y = 0$ from $(0,0)$ to $(2,0) \Rightarrow u = v$; the line segment $x + 2y = 2$ from $(\frac{2}{3}, \frac{2}{3})$ to $(2,0) \Rightarrow u = 2$. The transformed region is sketched at the right.



3. (a) $3x + 2y = u$ and $x + 4y = v \Rightarrow -5x = -2u + v$ and $y = \frac{1}{2}(u - 3x) \Rightarrow x = \frac{1}{5}(2u - v)$ and $y = \frac{1}{10}(3v - u)$;

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{vmatrix} = \frac{6}{50} - \frac{1}{50} = \frac{1}{10}$$

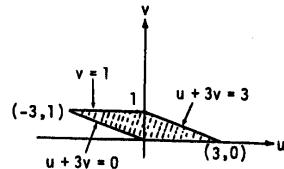
- (b) The x -axis $y = 0 \Rightarrow u = 3v$; the y -axis $x = 0 \Rightarrow v = 2u$;
 the line $x + y = 1 \Rightarrow \frac{1}{5}(2u - v) + \frac{1}{10}(3v - u) = 1$
 $\Rightarrow 2(2u - v) + (3v - u) = 10 \Rightarrow 3u + v = 10$. The
 transformed region is sketched at the right.



4. (a) $2x - 3y = u$ and $-x + y = v \Rightarrow -x = u + 3v$ and $y = v + x \Rightarrow x = -u - 3v$ and $y = -u - 2v$;

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1 & -3 \\ -1 & -2 \end{vmatrix} = 2 - 3 = -1$$

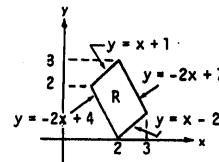
- (b) The line $x = -3 \Rightarrow -u - 3v = -3$ or $u + 3v = 3$;
 $x = 0 \Rightarrow u + 3v = 0$; $y = x \Rightarrow v = 0$; $y = x + 1 \Rightarrow v = 1$. The transformed region is the parallelogram sketched at the right.



$$\begin{aligned} 5. \int_0^4 \int_{y/2}^{(y/2)+1} \left(x - \frac{y}{2}\right) dx dy &= \int_0^4 \left[\frac{x^2}{2} - \frac{xy}{2}\right]_{y/2}^{(y/2)+1} dy = \frac{1}{2} \int_0^4 \left[\left(\frac{y}{2} + 1\right)^2 - \left(\frac{y}{2}\right)^2 - \left(\frac{y}{2} + 1\right)y + \left(\frac{y}{2}\right)y\right] dy \\ &= \frac{1}{2} \int_0^4 (y + 1 - y) dy = \frac{1}{2} \int_0^4 dy = \frac{1}{2}(4) = 2 \end{aligned}$$

$$\begin{aligned} 6. \iint_R (2x^2 - xy - y^2) dx dy &= \iint_R (x - y)(2x + y) dx dy \\ &= \iint_G uv \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv = \frac{1}{3} \iint_G uv du dv; \end{aligned}$$

We find the boundaries of G from the boundaries of R, shown in the accompanying figure:

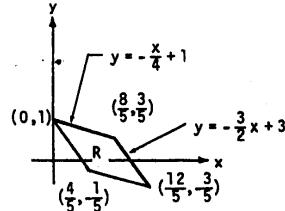


| xy-equations for the boundary of R | Corresponding uv-equations for the boundary of G | Simplified uv-equations |
|------------------------------------|--|-------------------------|
| $y = -2x + 4$ | $\frac{1}{3}(-2u + v) = -\frac{2}{3}(u + v) + 4$ | $v = 4$ |
| $y = -2x + 7$ | $\frac{1}{3}(-2u + v) = -\frac{2}{3}(u + v) + 7$ | $v = 7$ |
| $y = x - 2$ | $\frac{1}{3}(-2u + v) = \frac{1}{3}(u + v) - 2$ | $u = 2$ |
| $y = x + 1$ | $\frac{1}{3}(-2u + v) = \frac{1}{3}(u + v) + 1$ | $u = -1$ |

$$\Rightarrow \frac{1}{3} \int_G uv \, du \, dv = \frac{1}{3} \int_{-1}^2 \int_4^7 uv \, dv \, du = \frac{1}{3} \int_{-1}^2 u \left[\frac{v^2}{2} \right]_4^7 \, du = \frac{11}{2} \int_{-1}^2 u \, du = \left(\frac{11}{2} \right) \left[\frac{u^2}{2} \right]_{-1}^2 = \left(\frac{11}{4} \right) (4 - 1) = \frac{33}{4}$$

$$\begin{aligned} 7. \int_R (3x^2 + 14xy + 8y^2) \, dx \, dy &= \int_R (3x + 2y)(x + 4y) \, dx \, dy \\ &= \int_G uv \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du \, dv = \frac{1}{10} \int_G uv \, du \, dv; \end{aligned}$$

We find the boundaries of G from the boundaries of R, shown in the accompanying figure:



| xy-equations for the boundary of R | Corresponding uv-equations for the boundary of G | Simplified uv-equations |
|------------------------------------|--|-------------------------|
| $y = -\frac{3}{2}x + 1$ | $\frac{1}{10}(3v - u) = -\frac{3}{10}(2u - v) + 1$ | $u = 2$ |
| $y = -\frac{3}{2}x + 3$ | $\frac{1}{10}(3v - u) = -\frac{3}{10}(2u - v) + 3$ | $u = 6$ |
| $y = -\frac{1}{4}x$ | $\frac{1}{10}(3v - u) = -\frac{1}{20}(2u - v)$ | $v = 0$ |
| $y = -\frac{1}{4}x + 1$ | $\frac{1}{10}(3v - u) = -\frac{1}{20}(2u - v) + 1$ | $v = 4$ |

$$\Rightarrow \frac{1}{10} \int_G uv \, du \, dv = \frac{1}{10} \int_2^6 \int_0^4 uv \, dv \, du = \frac{1}{10} \int_2^6 u \left[\frac{v^2}{2} \right]_0^4 \, du = \frac{4}{5} \int_2^6 u \, du = \left(\frac{4}{5} \right) \left[\frac{u^2}{2} \right]_2^6 = \left(\frac{4}{5} \right) (18 - 2) = \frac{64}{5}$$

$$\begin{aligned} 8. \int_R 2(x - y) \, dx \, dy &= \int_G -2v \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du \, dv = \int_G -2v \, du \, dv; \text{ the region } G \text{ is sketched in Exercise 4} \\ &\Rightarrow \int_G -2v \, du \, dv = \int_0^1 \int_{-3v}^{3-v} -2v \, du \, dv = \int_0^1 -2v(3 - 3v + 3v) \, dv = \int_0^1 -6v \, dv = [-3v^2]_0^1 = -3 \end{aligned}$$

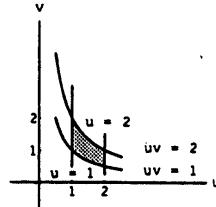
$$9. x = \frac{u}{v} \text{ and } y = uv \Rightarrow \frac{y}{x} = v^2 \text{ and } xy = u^2; \frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix} = v^{-1}u + v^{-1}u = \frac{2u}{v};$$

$y = x \Rightarrow uv = \frac{u}{v} \Rightarrow v = 1$, and $y = 4x \Rightarrow v = 2$; $xy = 1 \Rightarrow u = 1$, and $xy = 9 \Rightarrow u = 3$; thus

$$\begin{aligned} \int_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy} \right) dx dy &= \int_1^3 \int_1^2 (v+u)\left(\frac{2u}{v}\right) dv du = \int_1^3 \int_1^2 \left(2u + \frac{2u^2}{v}\right) dv du = \int_1^3 [2uv + 2u^2 \ln v]_1^2 du \\ &= \int_1^3 (2u + 2u^2 \ln 2) du = \left[u^2 + \frac{2}{3}u^3 \ln 2\right]_1^3 = 8 + \frac{2}{3}(26)(\ln 2) = 8 + \frac{52}{3}(\ln 2) \end{aligned}$$

$$10. (a) \frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u, \text{ and}$$

the region G is sketched at the right



(b) $x = 1 \Rightarrow u = 1$, and $x = 2 \Rightarrow u = 2$; $y = 1 \Rightarrow uv = 1 \Rightarrow v = \frac{1}{u}$, and $y = 2 \Rightarrow uv = 2 \Rightarrow v = \frac{2}{u}$; thus,

$$\begin{aligned} \int_1^2 \int_{1/u}^{2/u} \frac{y}{x} dy dx &= \int_1^2 \int_{1/u}^{2/u} \left(\frac{uv}{u}\right) u dv du = \int_1^2 \int_{1/u}^{2/u} uv dv du = \int_1^2 u \left[\frac{v^2}{2}\right]_{1/u}^{2/u} du = \int_1^2 u \left(\frac{2}{u^2} - \frac{1}{2u^2}\right) du \end{aligned}$$

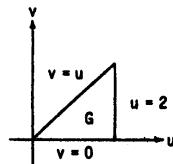
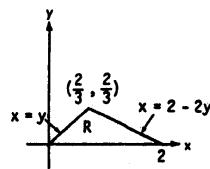
$$= \frac{3}{2} \int_1^2 u \left(\frac{1}{u^2}\right) du = \frac{3}{2} [\ln u]_1^2 = \frac{3}{2} \ln 2$$

$$11. x = ar \cos \theta \text{ and } y = br \sin \theta \Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = J(r,\theta) = \begin{vmatrix} a \cos \theta & -ar \sin \theta \\ b \sin \theta & br \cos \theta \end{vmatrix} = abr \cos^2 \theta + abr \sin^2 \theta = abr;$$

$$\begin{aligned} I_0 &= \int_R \int (x^2 + y^2) dA = \int_0^{2\pi} \int_0^1 r^2 (a^2 \cos^2 \theta + b^2 \sin^2 \theta) |J(r,\theta)| dr d\theta = \int_0^{2\pi} \int_0^1 abr^3 (a^2 \cos^2 \theta + b^2 \sin^2 \theta) dr d\theta \\ &= \frac{ab}{4} \int_0^{2\pi} (a^2 \cos^2 \theta + b^2 \sin^2 \theta) d\theta = \frac{ab}{4} \left[\frac{a^2 \theta}{2} + \frac{a^2 \sin 2\theta}{4} + \frac{b^2 \theta}{2} - \frac{b^2 \sin 2\theta}{4} \right]_0^{2\pi} = \frac{ab\pi(a^2 + b^2)}{4} \end{aligned}$$

$$\begin{aligned} 12. \frac{\partial(x,y)}{\partial(u,v)} = J(u,v) &= \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab; A = \int_R \int dy dx = \int_G ab du dv = \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab dv du \\ &= 2ab \int_{-1}^1 \sqrt{1-u^2} du = 2ab \left[\frac{u}{2} \sqrt{1-u^2} + \frac{1}{2} \sin^{-1} u \right]_{-1}^1 = ab [\sin^{-1} 1 - \sin^{-1} (-1)] = ab \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right] = ab\pi \end{aligned}$$

13. The region of integration R in the xy -plane is sketched in the figure at the right. The boundaries of the image G are obtained as follows, with G sketched at the right:



| xy-equations for the boundary of R | Corresponding uv-equations for the boundary of G | Simplified uv-equations |
|--------------------------------------|--|-------------------------|
| $x = y$ | $\frac{1}{3}(u + 2v) = \frac{1}{3}(u - v)$ | $v = 0$ |
| $x = 2 - 2y$ | $\frac{1}{3}(u + 2v) = 2 - \frac{2}{3}(u - v)$ | $u = 2$ |
| $y = 0$ | $0 = \frac{1}{3}(u - v)$ | $v = u$ |

Also, from Exercise 2, $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = -\frac{1}{3} \Rightarrow \int_0^{2/3} \int_y^{2-2y} (x+2y)e^{(y-x)} dx dy = \int_0^2 \int_0^u ue^{-v} \left| -\frac{1}{3} \right| dv du$

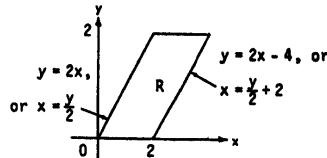
$$= \frac{1}{3} \int_0^2 u [-e^{-v}]_0^u du = \frac{1}{3} \int_0^2 u (1 - e^{-u}) du = \frac{1}{3} \left[u(u + e^{-u}) - \frac{u^2}{2} + e^{-u} \right]_0^2 = \frac{1}{3} [2(2 + e^{-2}) - 2 + e^{-2} - 1]$$

$$= \frac{1}{3}(3e^{-2} + 1) \approx 0.4687$$

14. $x = u + \frac{v}{2}$ and $y = v \Rightarrow 2x - y = (2u + v) - v = 2u$ and

$$\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = 1; \text{ next, } u = x - \frac{v}{2}$$

$= x - \frac{v}{2}$ and $v = y$, so the boundaries of the region of integration R in the xy -plane are transformed to the boundaries of G :



| xy-equations for the boundary of R | Corresponding uv-equations for the boundary of G | Simplified uv-equations |
|------------------------------------|--|-------------------------|
| $x = \frac{y}{2}$ | $u + \frac{v}{2} = \frac{v}{2}$ | $u = 0$ |
| $x = \frac{y}{2} + 2$ | $u + \frac{v}{2} = \frac{v}{2} + 2$ | $u = 2$ |
| $y = 0$ | $v = 0$ | $v = 0$ |
| $y = 2$ | $v = 2$ | $v = 2$ |

$$\Rightarrow \int_0^2 \int_{y/2}^{(y/2)+2} y^3(2x-y) e^{(2x-y)^2} dx dy = \int_0^2 \int_0^2 v^3(2u) e^{4u^2} du dv = \int_0^2 v^3 \left[\frac{1}{4} e^{4u^2} \right]_0^2 dv = \frac{1}{4} \int_0^2 v^3 (e^{16} - 1) dv \\ = \frac{1}{4} (e^{16} - 1) \left[\frac{v^4}{4} \right]_0^2 = e^{16} - 1$$

15. (a) $x = u \cos v$ and $y = u \sin v \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \end{vmatrix} = u \cos^2 v + u \sin^2 v = u$

(b) $x = u \sin v$ and $y = u \cos v \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sin v & u \cos v \\ \cos v & -u \sin v \end{vmatrix} = -u \sin^2 v - u \cos^2 v = -u$

16. (a) $x = u \cos v, y = u \sin v, z = w \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \cos v & -u \sin v & 0 \\ \sin v & u \cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u \cos^2 v + u \sin^2 v = u$

(b) $x = 2u - 1, y = 3v - 4, z = \frac{1}{2}(w - 4) \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = (2)(3)\left(\frac{1}{2}\right) = 3$

17. $\begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix}$

$$= (\cos \phi) \begin{vmatrix} \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \end{vmatrix} + (\rho \sin \phi) \begin{vmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta \end{vmatrix}$$

$$= (\rho^2 \cos \phi)(\sin \phi \cos \phi \cos^2 \theta + \sin \phi \cos \phi \sin^2 \theta) + (\rho^2 \sin \phi)(\sin^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta)$$

$$= \rho^2 \sin \phi \cos^2 \phi + \rho^2 \sin^3 \phi = (\rho^2 \sin \phi)(\cos^2 \phi + \sin^2 \phi) = \rho^2 \sin \phi$$

18. Let $u = g(x) \Rightarrow J(x) = \frac{du}{dx} = g'(x) \Rightarrow \int_a^b f(u) du = \int_{g(a)}^{g(b)} f(g(x))g'(x) dx$ in accordance with formula (1) in

Section 4.8. Note that $g'(x)$ represents the Jacobian of the transformation $u = g(x)$ or $x = g^{-1}(u)$. Several examples are presented in Section 4.6.

$$\begin{aligned} 19. \int_0^3 \int_0^4 \int_{y/2}^{1+(y/2)} & \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx dy dz = \int_0^3 \int_0^4 \left[\frac{x^2}{2} - \frac{xy}{2} + \frac{xz}{3} \right]_{y/2}^{1+(y/2)} dy dz = \int_0^3 \int_0^4 \left[\frac{1}{2}(y+1) - \frac{y}{2} + \frac{z}{3} \right] dy dz \\ &= \int_0^3 \left[\frac{(y+1)^2}{4} - \frac{y^2}{4} + \frac{yz}{3} \right]_0^4 dz = \int_0^3 \left(\frac{9}{4} + \frac{4z}{3} - \frac{1}{4} \right) dz = \int_0^3 \left(2 + \frac{4z}{3} \right) dz = \left[2z + \frac{2z^2}{3} \right]_0^3 = 12 \end{aligned}$$

$$20. J(u, v, w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc; \text{ the transformation takes the ellipsoid region } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \text{ in xyz-space}$$

into the spherical region $u^2 + v^2 + w^2 \leq 1$ in uvw-space (which has volume $V = \frac{4}{3}\pi$)

$$\Rightarrow V = \int_R \int \int dx dy dz = \int_G \int \int abc du dv dw = \frac{4\pi abc}{3}$$

$$\begin{aligned} 21. J(u, v, w) &= \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc; \text{ for R and G as in Exercise 20, } \int_R \int \int |xyz| dx dy dz \\ &= \int_G \int \int a^2 b^2 c^2 uvw dw dv du = 8a^2 b^2 c^2 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 (\rho \sin \phi \cos \theta)(\rho \sin \phi \sin \theta)(\rho \cos \phi)(\rho^2 \sin \phi) d\rho d\phi d\theta \\ &= \frac{4a^2 b^2 c^2}{3} \int_0^{\pi/2} \int_0^{\pi/2} \sin \theta \cos \theta \sin^3 \phi \cos \phi d\phi d\theta = \frac{a^2 b^2 c^2}{3} \int_0^{\pi/2} \sin \theta \cos \theta d\theta = \frac{a^2 b^2 c^2}{6} \end{aligned}$$

$$\begin{aligned} 22. u = x, v = xy, \text{ and } w = 3z \Rightarrow x = u, y = \frac{v}{u}, \text{ and } z = \frac{1}{3}w \Rightarrow J(u, v, w) &= \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u}; \\ \int_R \int \int (x^2 y + 3xyz) dx dy dz &= \int_G \int \int \left[u^2 \left(\frac{v}{u} \right) + 3u \left(\frac{v}{u} \right) \left(\frac{w}{3} \right) \right] |J(u, v, w)| du dv dw = \frac{1}{3} \int_0^3 \int_0^2 \int_1^2 \left(v + \frac{vw}{u} \right) du dv dw \\ &= \frac{1}{3} \int_0^3 \int_0^2 \left(v + vw \ln 2 \right) dv dw = \frac{1}{3} \int_0^3 \left(1 + w \ln 2 \right) \left[\frac{v^2}{2} \right]_0^2 dw = \frac{2}{3} \int_0^3 \left(1 + w \ln 2 \right) dw = \frac{2}{3} \left[w + \frac{w^2}{2} \ln 2 \right]_0^3 \end{aligned}$$

$$= \frac{2}{3} \left(3 + \frac{9}{2} \ln 2 \right) = 2 + 3 \ln 2 = 2 + \ln 8$$

23. The first moment about the xy -coordinate plane for the semi-ellipsoid, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ using the

$$\text{transformation in Exercise 21 is, } M_{xy} = \int_D \int z \, dz \, dy \, dx = \int_G \int c w |J(u, v, w)| \, du \, dv \, dw$$

$$= abc^2 \int_G w \, du \, dv \, dw = (abc^2) \cdot (M_{xy} \text{ of the hemisphere } x^2 + y^2 + z^2 = 1, z \geq 0) = \frac{abc^2 \pi}{4};$$

$$\text{the mass of the semi-ellipsoid is } \frac{2abc\pi}{3} \Rightarrow \bar{z} = \left(\frac{abc^2 \pi}{4} \right) \left(\frac{3}{2abc\pi} \right) = \frac{3}{8}c$$

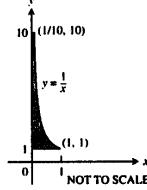
24. A solid of revolution is symmetric about the axis of revolution, therefore, the height of the solid is solely a function of r . That is, $y = f(x) = f(r)$. Using cylindrical coordinates with $x = r \cos \theta$, $y = r \sin \theta$ and $z = r$ we have

$$\text{we have } V = \int_G \int r \, dy \, d\theta \, dr = \int_a^b \int_0^{2\pi} \int_0^{f(r)} r \, dy \, d\theta \, dr = \int_a^b \int_0^{2\pi} [ry]_0^{f(r)} \, d\theta \, dr = \int_a^b \int_0^{2\pi} r f(r) \, d\theta \, dr = \int_a^b [r\theta f(r)]_0^{2\pi} \, dr$$

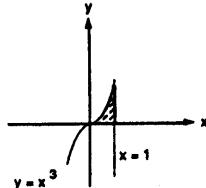
$= \int_a^b 2\pi r f(r) \, dr$. In the last integral, r is a dummy or stand-in variable and as such it can be replaced by any variable name. Choosing x instead of r we have $V = \int_a^b 2\pi x f(x) \, dx$, which is the same result obtained using the shell method.

CHAPTER 12 PRACTICE EXERCISES

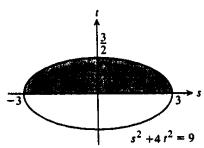
$$\begin{aligned} 1. \quad & \int_1^{10} \int_0^{1/y} ye^{xy} \, dx \, dy = \int_1^{10} [e^{xy}]_0^{1/y} \, dy \\ &= \int_1^{10} (e - 1) \, dy = 9e - 9 \end{aligned}$$



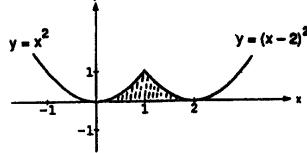
$$\begin{aligned} 2. \quad & \int_0^1 \int_0^{x^3} e^{y/x} \, dy \, dx = \int_0^1 x [e^{y/x}]_0^{x^3} \, dx \\ &= \int_0^1 (xe^{x^2} - x) \, dx = \left[\frac{1}{2}e^{x^2} - \frac{x^2}{2} \right]_0^1 = \frac{e-2}{2} \end{aligned}$$



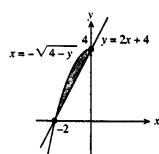
$$\begin{aligned}
 3. \quad & \int_0^{3/2} \int_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} t \, ds \, dt = \int_0^{3/2} [ts] \Big|_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} \, dt \\
 &= \int_0^{3/2} 2t\sqrt{9-4t^2} \, dt = \left[-\frac{1}{6}(9-4t^2)^{3/2} \right]_0^{3/2} \\
 &= -\frac{1}{6}(0^{3/2} - 9^{3/2}) = \frac{27}{6} = \frac{9}{2}
 \end{aligned}$$



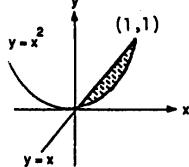
$$\begin{aligned}
 4. \quad & \int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} xy \, dx \, dy = \int_0^1 y \left[\frac{x^2}{2} \right]_{\sqrt{y}}^{2-\sqrt{y}} \, dy \\
 &= \frac{1}{2} \int_0^1 y (4 - 4\sqrt{y} + y - y) \, dy \\
 &= \int_0^1 (2y - 2y^{3/2}) \, dy = \left[y^2 - \frac{4y^{5/2}}{5} \right]_0^1 = \frac{1}{5}
 \end{aligned}$$



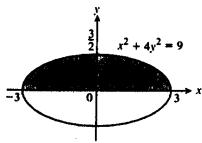
$$\begin{aligned}
 5. \quad & \int_{-2}^0 \int_{2x+4}^{4-x^2} dy \, dx = \int_{-2}^0 (-x^2 - 2x) \, dx \\
 &= \left[-\frac{x^3}{3} - x^2 \right]_{-2}^0 = -\left(\frac{8}{3} - 4 \right) = \frac{4}{3}
 \end{aligned}$$



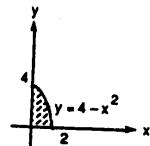
$$\begin{aligned}
 6. \quad & \int_0^1 \int_y^{\sqrt{y}} \sqrt{x} \, dx \, dy = \int_0^1 \left[\frac{2}{3}x^{3/2} \right]_y^{\sqrt{y}} \, dy \\
 &= \frac{2}{3} \int_0^1 (y^{3/4} - y^{3/2}) \, dy = \frac{2}{3} \left[\frac{4}{7}y^{7/4} - \frac{2}{5}y^{5/2} \right]_0^1 \\
 &= \frac{2}{3} \left(\frac{4}{7} - \frac{2}{5} \right) = \frac{4}{35}
 \end{aligned}$$



$$\begin{aligned}
 7. \int_{-3}^3 \int_0^{(1/2)\sqrt{9-x^2}} y \, dy \, dx &= \int_{-3}^3 \left[\frac{y^2}{2} \right]_0^{(1/2)\sqrt{9-x^2}} \, dx \\
 &= \int_{-3}^3 \frac{1}{8}(9-x^2) \, dx = \left[\frac{9x}{8} - \frac{x^3}{24} \right]_0^3 \\
 &= \left(\frac{27}{8} - \frac{27}{24} \right) - \left(-\frac{27}{8} + \frac{27}{24} \right) = \frac{27}{6} = \frac{9}{2}
 \end{aligned}$$



$$\begin{aligned}
 8. \int_0^4 \int_0^{\sqrt{4-y}} 2x \, dx \, dy &= \int_0^4 \left[x^2 \right]_0^{\sqrt{4-y}} \, dy \\
 &= \int_0^4 (4-y) \, dy = \left[4y - \frac{y^2}{2} \right]_0^4 = 8
 \end{aligned}$$



$$9. \int_0^1 \int_{2y}^2 4 \cos(x^2) \, dx \, dy = \int_0^2 \int_0^{\pi/2} 4 \cos(x^2) \, dy \, dx = \int_0^2 2x \cos(x^2) \, dx = [\sin(x^2)]_0^2 = \sin 4$$

$$10. \int_0^1 \int_{y/2}^1 e^{x^2} \, dx \, dy = \int_0^1 \int_0^{2x} e^{x^2} \, dy \, dx = \int_0^1 2xe^{x^2} \, dx = [e^{x^2}]_0^1 = e - 1$$

$$11. \int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1} \, dy \, dx = \int_0^2 \int_0^{y^3} \frac{1}{y^4+1} \, dx \, dy = \frac{1}{4} \int_0^2 \frac{4y^3}{y^4+1} \, dy = \frac{\ln 17}{4}$$

$$12. \int_0^1 \int_{\sqrt[3]{y}}^1 \frac{2\pi \sin(\pi x^2)}{x^2} \, dx \, dy = \int_0^1 \int_0^{x^3} \frac{2\pi \sin(\pi x^2)}{x^2} \, dy \, dx = \int_0^1 2\pi x \sin(\pi x^2) \, dx = [-\cos(\pi x^2)]_0^1 = -(-1) - (-1) = 2$$

$$13. A = \int_{-2}^0 \int_{2x+4}^{4-x^2} dy \, dx = \int_{-2}^0 (-x^2 - 2x) \, dx = \frac{4}{3} \quad 14. A = \int_1^4 \int_{2-y}^{\sqrt{y}} dx \, dy = \int_1^4 (\sqrt{y} - 2 + y) \, dy = \frac{37}{6}$$

$$\begin{aligned}
 15. V &= \int_0^1 \int_x^{2-x} (x^2 + y^2) \, dy \, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_x^{2-x} \, dx = \int_0^1 \left[2x^2 + \frac{(2-x)^3}{3} - \frac{7x^3}{3} \right] \, dx = \left[\frac{2x^3}{3} - \frac{(2-x)^4}{12} - \frac{7x^4}{12} \right]_0^1 \\
 &= \left(\frac{2}{3} - \frac{1}{12} - \frac{7}{12} \right) + \frac{2^4}{3} = \frac{4}{3}
 \end{aligned}$$

$$16. V = \int_{-3}^2 \int_x^{6-x^2} x^2 dy dx = \int_{-3}^2 [x^2 y]_x^{6-x^2} dx = \int_{-3}^2 (6x^2 - x^4 - x^3) dx = \frac{125}{4}$$

$$17. \text{ average value} = \int_0^1 \int_0^1 xy dy dx = \int_0^1 \left[\frac{xy^2}{2} \right]_0^1 dx = \int_0^1 \frac{x}{2} dx = \frac{1}{4}$$

$$18. \text{ average value} = \left(\frac{1}{\pi} \right) \int_0^1 \int_0^{\sqrt{1-x^2}} xy dy dx = \frac{4}{\pi} \int_0^1 \left[\frac{xy^2}{2} \right]_0^{\sqrt{1-x^2}} dx = \frac{2}{\pi} \int_0^1 (x - x^3) dx = \frac{1}{2\pi}$$

$$19. M = \int_1^2 \int_{2/x}^2 dy dx = \int_1^2 \left(2 - \frac{2}{x} \right) dx = 2 - \ln 4; M_y = \int_1^2 \int_{2/x}^2 x dy dx = \int_1^2 x \left(2 - \frac{2}{x} \right) dx = 1;$$

$$M_x = \int_1^2 \int_{2/x}^2 y dy dx = \int_1^2 \left(2 - \frac{2}{x^2} \right) dx = 1 \Rightarrow \bar{x} = \bar{y} = \frac{1}{2 - \ln 4}$$

$$20. M = \int_0^4 \int_{-2y}^{2y-y^2} dx dy = \int_0^4 (4y - y^2) dy = \frac{32}{3}; M_x = \int_0^4 \int_{-2y}^{2y-y^2} y dx dy = \int_0^4 (4y^2 - y^3) dy = \left[\frac{4y^3}{3} - \frac{y^4}{4} \right]_0^4 = \frac{64}{3};$$

$$M_y = \int_0^4 \int_{-2y}^{2y-y^2} x dx dy = \int_0^4 \left[\frac{(2y-y^2)^2}{2} - 2y^2 \right] dy = \left[\frac{y^5}{10} - \frac{y^4}{2} \right]_0^4 = -\frac{128}{5} \Rightarrow \bar{x} = \frac{M_y}{M} = -\frac{12}{5} \text{ and } \bar{y} = \frac{M_x}{M} = 2$$

$$21. I_o = \int_0^2 \int_{2x}^4 (x^2 + y^2)(3) dy dx = 3 \int_0^2 \left(4x^2 + \frac{64}{3} - \frac{14x^3}{3} \right) dx = 104$$

$$22. (a) I_o = \int_{-2}^2 \int_{-1}^1 (x^2 + y^2) dy dx = \int_{-2}^2 \left(2x^2 + \frac{2}{3} \right) dx = \frac{40}{3}$$

$$(b) I_x = \int_{-a}^a \int_{-b}^b y^2 dy dx = \int_{-a}^a \frac{2b^3}{3} dx = \frac{4ab^3}{3}; I_y = \int_{-a}^a \int_{-b}^b x^2 dx dy = \int_{-b}^b \frac{2a^3}{3} dy = \frac{4a^3b}{3} \Rightarrow I_o = I_x + I_y$$

$$= \frac{4ab^3}{3} + \frac{4a^3b}{3} = \frac{4ab(b^2 + a^2)}{3}$$

$$23. M = \delta \int_0^3 \int_0^{2x/3} dy dx = \delta \int_0^3 \frac{2x}{3} dx = 3\delta; I_x = \delta \int_0^3 \int_0^{2x/3} y^2 dy dx = \frac{8\delta}{81} \int_0^3 x^3 dx = \left(\frac{8\delta}{81} \right) \left(\frac{3^4}{4} \right) = 2\delta \Rightarrow R_x = \sqrt{\frac{2}{3}}$$

$$24. M = \int_0^1 \int_{x^2}^x (x+1) dy dx = \int_0^1 (x - x^3) dx = \frac{1}{4}; M_x = \int_0^1 \int_{x^2}^x y(x+1) dy dx = \frac{1}{2} \int_0^1 (x^3 - x^5 + x^2 - x^4) dx = \frac{13}{120};$$

$$\begin{aligned}
M_y &= \int_0^1 \int_{x^2}^x x(x+1) dy dx = \int_0^1 (x^2 - x^4) dx = \frac{2}{15} \Rightarrow \bar{x} = \frac{8}{15} \text{ and } \bar{y} = \frac{13}{30}; I_x = \int_0^1 \int_{x^2}^x y^2(x+1) dy dx \\
&= \frac{1}{3} \int_0^1 (x^4 - x^7 + x^3 - x^6) dx = \frac{17}{280} \Rightarrow R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{17}{70}}; I_y = \int_0^1 \int_{x^2}^x x^2(x+1) dy dx = \int_0^1 (x^3 - x^5) dx \\
&= \frac{1}{12} \Rightarrow R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{1}{3}}
\end{aligned}$$

$$\begin{aligned}
25. M &= \int_{-1}^1 \int_{-1}^1 \left(x^2 + y^2 + \frac{1}{3}\right) dy dx = \int_{-1}^1 \left(2x^2 + \frac{4}{3}\right) dx = 4; M_x = \int_{-1}^1 \int_{-1}^1 y \left(x^2 + y^2 + \frac{1}{3}\right) dy dx = \int_{-1}^1 0 dx = 0; \\
M_y &= \int_{-1}^1 \int_{-1}^1 x \left(x^2 + y^2 + \frac{1}{3}\right) dy dx = \int_{-1}^1 \left(2x^3 + \frac{4}{3}x\right) dx = 0
\end{aligned}$$

26. Place the ΔABC with its vertices at $A(0,0)$, $B(b,0)$ and $C(a,h)$. The line through the points A and C is

$$\begin{aligned}
y &= \frac{h}{a}x; \text{ the line through the points C and B is } y = \frac{h}{a-b}(x-b). \text{ Thus, } M = \int_0^h \int_{ay/h}^{(a-b)y/h+b} \delta dx dy \\
&= b\delta \int_0^h \left(1 - \frac{y}{h}\right) dy = \frac{\delta bh}{2}; I_x = \int_0^h \int_{ay/h}^{(a-b)y/h+b} y^2 \delta dx dy = b\delta \int_0^h \left(y^2 - \frac{y^3}{h}\right) dy = \frac{\delta b h^3}{12}; R_x = \sqrt{\frac{I_x}{M}} = \frac{h}{\sqrt{6}}
\end{aligned}$$

$$27. \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2} dy dx = \int_0^{2\pi} \int_0^1 \frac{2r}{(1+r^2)^2} dr d\theta = \int_0^{2\pi} \left[-\frac{1}{1+r^2}\right]_0^1 d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \pi$$

$$\begin{aligned}
28. \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1) dx dy &= \int_0^{2\pi} \int_0^1 r \ln(r^2+1) dr d\theta = \int_0^{2\pi} \int_1^2 \frac{1}{2} \ln u du d\theta = \frac{1}{2} \int_0^{2\pi} [u \ln u - u]_1^2 d\theta \\
&= \frac{1}{2} \int_0^{2\pi} (2 \ln 2 - 1) d\theta = [\ln(4) - 1]\pi
\end{aligned}$$

$$29. M = \int_{-\pi/3}^{\pi/3} \int_0^3 r dr d\theta = \frac{9}{2} \int_{-\pi/3}^{\pi/3} d\theta = 3\pi; M_y = \int_{-\pi/3}^{\pi/3} \int_0^3 r^2 \cos \theta dr d\theta = 9 \int_{-\pi/3}^{\pi/3} \cos \theta d\theta = 9\sqrt{3} \Rightarrow \bar{x} = \frac{3\sqrt{3}}{\pi},$$

and $\bar{y} = 0$ by symmetry

$$30. M = \int_0^{\pi/2} \int_1^3 r dr d\theta = 4 \int_0^{\pi/2} d\theta = 2\pi; M_y = \int_0^{\pi/2} \int_1^3 r^2 \cos \theta dr d\theta = \frac{26}{3} \int_0^{\pi/2} \cos \theta d\theta = \frac{26}{3} \Rightarrow \bar{x} = \frac{13}{3\pi}, \text{ and}$$

$$\bar{y} = \frac{13}{3\pi} \text{ by symmetry}$$

31. (a) $M = 2 \int_0^{\pi/2} \int_1^{1+\cos\theta} r dr d\theta$

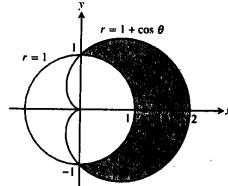
$$= \int_0^{\pi/2} \left(2 \cos\theta + \frac{1 + \cos 2\theta}{2} \right) d\theta = \frac{8 + \pi}{4};$$

$$M_y = \int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} (r \cos\theta) r dr d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\cos^2\theta + \cos^3\theta + \frac{\cos^4\theta}{3} \right) d\theta$$

$$= \frac{32 + 15\pi}{24} \Rightarrow \bar{x} = \frac{15\pi + 32}{6\pi + 48}, \text{ and}$$

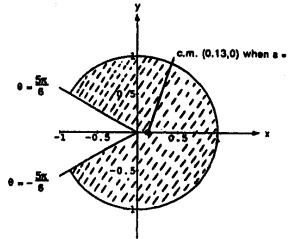
$$\bar{y} = 0 \text{ by symmetry}$$



32. (a) $M = \int_{-\alpha}^{\alpha} \int_0^a r dr d\theta = \int_{-\alpha}^{\alpha} \frac{a^2}{2} d\theta = a^2\alpha; M_y = \int_{-\alpha}^{\alpha} \int_0^a (r \cos\theta) r dr d\theta = \int_{-\alpha}^{\alpha} \frac{a^3 \cos\theta}{3} d\theta = \frac{2a^3 \sin\alpha}{3}$

$$\Rightarrow \bar{x} = \frac{2a \sin\alpha}{3\alpha}, \text{ and } \bar{y} = 0 \text{ by symmetry; } \lim_{\alpha \rightarrow \pi^-} \bar{x} = \lim_{\alpha \rightarrow \pi^-} \frac{2a \sin\alpha}{3\alpha} = 0$$

(b) $\bar{x} = \frac{2a}{5\pi}$ and $\bar{y} = 0$



33. $(x^2 + y^2)^2 - (x^2 - y^2) = 0 \Rightarrow r^4 - r^2 \cos 2\theta = 0 \Rightarrow r^2 = \cos 2\theta$ so the integral is

$$\int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2} dr d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left[-\frac{1}{2(1+r^2)} \right]_0^{\sqrt{\cos 2\theta}} d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(1 - \frac{1}{1+\cos 2\theta} \right) d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(1 - \frac{1}{2 \cos^2 \theta} \right) d\theta$$

$$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(1 - \frac{\sec^2 \theta}{2} \right) d\theta = \frac{1}{2} \left[\theta - \frac{\tan \theta}{2} \right]_{-\pi/4}^{\pi/4} = \frac{\pi - 2}{4}$$

$$\begin{aligned}
 34. \text{ (a)} \int_R \int \frac{1}{(1+x^2+y^2)^2} dx dy &= \int_0^{\pi/3} \int_0^{\sec \theta} \frac{r}{(1+r^2)^2} dr d\theta = \int_0^{\pi/3} \left[-\frac{1}{2(1+r^2)} \right]_0^{\sec \theta} d\theta \\
 &= \int_0^{\pi/3} \left[\frac{1}{2} - \frac{1}{2(1+\sec^2 \theta)} \right] d\theta = \frac{1}{2} \int_0^{\pi/3} \frac{\sec^2 \theta}{1+\sec^2 \theta} d\theta; \begin{cases} u = \tan \theta \\ du = \sec^2 \theta d\theta \end{cases} \rightarrow \frac{1}{2} \int_0^{\sqrt{3}} \frac{du}{2+u^2} \\
 &= \frac{1}{2} \left[\frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} \right]_0^{\sqrt{3}} = \frac{\sqrt{2}}{4} \tan^{-1} \sqrt{\frac{3}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \int_R \int \frac{1}{(1+x^2+y^2)^2} dx dy &= \int_0^{\pi/2} \int_0^\infty \frac{r}{(1+r^2)^2} dr d\theta = \int_0^{\pi/2} \lim_{b \rightarrow \infty} \left[-\frac{1}{2(1+r^2)} \right] d\theta \\
 &= \int_0^{\pi/2} \lim_{b \rightarrow \infty} \left[\frac{1}{2} - \frac{1}{2(1+b^2)} \right] d\theta = \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 35. \int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \cos(x+y+z) dx dy dz &= \int_0^{\pi} \int_0^{\pi} [\sin(z+y+\pi) - \sin(z+y)] dy dz \\
 &= \int_0^{\pi} [-\cos(z+2\pi) + \cos(z+\pi) + \cos z - \cos(z+\pi)] dz = 0
 \end{aligned}$$

$$36. \int_{\ln 6}^{\ln 7} \int_0^{\ln 2} \int_{\ln 4}^{\ln 5} e^{(x+y+z)} dz dy dx = \int_{\ln 6}^{\ln 7} \int_0^{\ln 2} e^{(x+y)} dy dx = \int_{\ln 6}^{\ln 7} e^x dx = 1$$

$$37. \int_0^1 \int_0^{x^2} \int_0^{x+y} (2x-y-z) dz dy dx = \int_0^1 \int_0^{x^2} \left(\frac{3x^2}{2} - \frac{3y^2}{2} \right) dy dx = \int_0^1 \left(\frac{3x^4}{2} - \frac{x^6}{2} \right) dx = \frac{8}{35}$$

$$38. \int_1^e \int_1^x \int_0^z \frac{2y}{z^3} dy dz dx = \int_1^e \int_1^x \frac{1}{z} dz dx = \int_1^e \ln x dx = [x \ln x - x]_1^e = 1$$

$$39. V = 2 \int_0^{\pi/2} \int_{-\cos y}^0 \int_0^{-2x} dz dx dy = 2 \int_0^{\pi/2} \int_{-\cos y}^0 -2x dx dy = 2 \int_0^{\pi/2} \cos^2 y dy = 2 \left[\frac{y}{2} + \frac{\sin 2y}{4} \right]_0^{\pi/2} = \frac{\pi}{2}$$

$$\begin{aligned}
 40. V &= 4 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{4-x^2} dz dy dx = 4 \int_0^2 \int_0^{\sqrt{4-x^2}} (4-x^2) dy dx = 4 \int_0^2 (4-x^2)^{3/2} dx \\
 &= \left[x(4-x^2)^{3/2} + 6x\sqrt{4-x^2} + 24 \sin^{-1} \frac{x}{2} \right]_0^2 = 24 \sin^{-1} 1 = 12\pi
 \end{aligned}$$

$$\begin{aligned}
 41. \text{ average} &= \frac{1}{3} \int_0^1 \int_0^3 \int_0^1 30xz\sqrt{x^2+y} \, dz \, dy \, dx = \frac{1}{3} \int_0^1 \int_0^3 15x\sqrt{x^2+y} \, dy \, dx = \frac{1}{3} \int_0^3 \int_0^1 15x\sqrt{x^2+y} \, dx \, dy \\
 &= \frac{1}{3} \int_0^3 \left[5(x^2+y)^{3/2} \right]_0^1 dy = \frac{1}{3} \int_0^3 [5(1+y)^{3/2} - 5y^{3/2}] dy = \frac{1}{3} \left[2(1+y)^{5/2} - 2y^{5/2} \right]_0^3 = \frac{1}{3} [2(4)^{5/2} - 2(3)^{5/2} - 2] \\
 &= \frac{1}{3} [2(31 - 3^{5/2})]
 \end{aligned}$$

$$42. \text{ average} = \frac{3}{4\pi a^3} \int_0^{2\pi} \int_0^\pi \int_0^a \rho^3 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{3a}{16\pi} \int_0^{2\pi} \int_0^\pi \sin \phi \, d\phi \, d\theta = \frac{3a}{8\pi} \int_0^{2\pi} d\theta = \frac{3a}{4}$$

$$43. \text{(a)} \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 3 \, dz \, dx \, dy$$

$$\text{(b)} \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 3\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$\begin{aligned}
 \text{(c)} \int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} 3 \, dz \, r \, dr \, d\theta &= 3 \int_0^{2\pi} \int_0^{\sqrt{2}} [r(4-r^2)^{1/2} - r^2] \, dr \, d\theta = 3 \int_0^{2\pi} \left[-\frac{1}{3}(4-r^2)^{3/2} - \frac{r^3}{3} \right]_0^{\sqrt{2}} \, d\theta \\
 &= \int_0^{2\pi} (-2^{3/2} - 2^{3/2} + 4^{3/2}) \, d\theta = (8 - 4\sqrt{2}) \int_0^{2\pi} \, d\theta = 2\pi(8 - 4\sqrt{2})
 \end{aligned}$$

$$44. \text{(a)} \int_{-\pi/2}^{\pi/2} \int_0^1 \int_{-r^2}^{r^2} 21(r \cos \theta)(r \sin \theta)^2 \, dz \, r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^1 \int_{-r^2}^{r^2} 21r^3 \cos \theta \sin^2 \theta \, dz \, r \, dr \, d\theta$$

$$\text{(b)} \int_{-\pi/2}^{\pi/2} \int_0^1 \int_{-r^2}^{r^2} 21r^3 \cos \theta \sin^2 \theta \, dz \, r \, dr \, d\theta = 84 \int_0^{\pi/2} \int_0^1 \int_{-r^2}^{r^2} r^6 \sin^2 \theta \cos \theta \, dr \, d\theta = 12 \int_0^{\pi/2} \sin^2 \theta \cos \theta \, d\theta = 4$$

$$45. \text{(a)} \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$\text{(b)} \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/4} (\sec \phi)(\sec \phi \tan \phi) \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \left[\frac{1}{2} \tan^2 \phi \right]_0^{\pi/4} \, d\theta = \frac{1}{6} \int_0^{2\pi} \, d\theta = \frac{\pi}{3}$$

$$46. \text{(a)} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} (6+4y) \, dz \, dy \, dx \quad \text{(b)} \int_0^{\pi/2} \int_0^1 \int_0^r (6+4r \sin \theta) \, dz \, r \, dr \, d\theta$$

$$(c) \int_0^{\pi/2} \int_{\pi/4}^{\pi/2} \int_0^{\csc \phi} (6 + 4\rho \sin \phi \sin \theta)(\rho^2 \sin \phi) d\rho d\phi d\theta$$

$$(d) \int_0^{\pi/2} \int_0^1 \int_0^r (6 + 4r \sin \theta) dz dr d\theta = \int_0^{\pi/2} \int_0^1 (6r^2 + 4r^3 \sin \theta) dr d\theta = \int_0^{\pi/2} [2r^3 + r^4 \sin \theta]_0^1 d\theta \\ = \int_0^{\pi/2} (2 + \sin \theta) d\theta = [2\theta - \cos \theta]_0^{\pi/2} = \pi + 1$$

$$47. \int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} z^2 yx dz dy dx + \int_1^{\sqrt{3}} \int_0^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} z^2 yx dz dy dx$$

48. (a) Bounded on the top and bottom by the sphere $x^2 + y^2 + z^2 = 4$, on the right by the right circular cylinder $(x-1)^2 + y^2 = 1$, on the left by the plane $y = 0$

$$(b) \int_0^{\pi/2} \int_0^{2 \cos \theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz r dr d\theta$$

$$49. (a) V = \int_0^{2\pi} \int_0^2 \int_2^{\sqrt{8-r^2}} dz r dr d\theta = \int_0^{2\pi} \int_0^2 (r\sqrt{8-r^2} - 2r) dr d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(8-r^2)^{3/2} - r^2 \right]_0^2 d\theta \\ = \int_0^{2\pi} \left[-\frac{1}{3}(4)^{3/2} - 4 + \frac{1}{3}(8)^{3/2} \right] d\theta = \int_0^{2\pi} \frac{4}{3}(-2 - 3 + 2\sqrt{8}) d\theta = \frac{4}{3}(4\sqrt{2} - 5) \int_0^{2\pi} d\theta = \frac{8\pi(4\sqrt{2} - 5)}{3}$$

$$(b) V = \int_0^{2\pi} \int_0^{\pi/4} \int_{2 \sec \phi}^{\sqrt{8}} \rho^2 \sin \phi d\rho d\phi d\theta = \frac{8}{3} \int_0^{2\pi} \int_0^{\pi/4} (2\sqrt{2} \sin \phi - \sec^3 \phi \sin \phi) d\phi d\theta \\ = \frac{8}{3} \int_0^{2\pi} \int_0^{\pi/4} (2\sqrt{2} \sin \phi - \tan \phi \sec^2 \phi) d\phi d\theta = \frac{8}{3} \int_0^{2\pi} \left[-2\sqrt{2} \cos \phi - \frac{1}{2} \tan^2 \phi \right]_0^{\pi/4} d\theta \\ = \frac{8}{3} \int_0^{2\pi} \left(-2 - \frac{1}{2} + 2\sqrt{2} \right) d\theta = \frac{8}{3} \int_0^{2\pi} \left(\frac{-5 + 4\sqrt{2}}{2} \right) d\theta = \frac{8\pi(4\sqrt{2} - 5)}{3}$$

$$50. I_z = \int_0^{2\pi} \int_0^{\pi/3} \int_0^2 (\rho \sin \phi)^2 (\rho^2 \sin \phi) d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^2 \rho^4 \sin^3 \phi d\rho d\phi d\theta \\ = \frac{32}{5} \int_0^{2\pi} \int_0^{\pi/3} (\sin \phi - \cos^2 \phi \sin \phi) d\phi d\theta = \frac{32}{5} \int_0^{2\pi} \left[-\cos \phi + \frac{\cos^3 \phi}{3} \right]_0^{\pi/3} d\theta = \frac{8\pi}{3}$$

51. With the centers of the spheres at the origin, $I_z = \int_0^{2\pi} \int_0^{\pi} \int_a^b \delta(\rho \sin \phi)^2 (\rho^2 \sin \phi) d\rho d\phi d\theta$

$$= \frac{\delta(b^5 - a^5)}{5} \int_0^{2\pi} \int_0^{\pi} \sin^3 \phi d\phi d\theta = \frac{\delta(b^5 - a^5)}{5} \int_0^{2\pi} \int_0^{\pi} (\sin \phi - \cos^2 \phi \sin \phi) d\phi d\theta$$

$$= \frac{\delta(b^5 - a^5)}{5} \int_0^{2\pi} \left[-\cos \phi + \frac{\cos^3 \phi}{3} \right]_0^{\pi} d\theta = \frac{4\delta(b^5 - a^5)}{15} \int_0^{2\pi} d\theta = \frac{8\pi\delta(b^5 - a^5)}{15}$$

52. $I_z = \int_0^{2\pi} \int_0^{\pi} \int_0^{1-\cos \theta} (\rho \sin \phi)^2 (\rho^2 \sin \phi) d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\pi} \int_0^{1-\cos \theta} \rho^4 \sin^3 \phi d\rho d\phi d\theta$

$$= \frac{1}{5} \int_0^{2\pi} \int_0^{\pi} (1 - \cos \phi)^5 \sin^3 \phi d\phi d\theta = \int_0^{2\pi} \int_0^{\pi} (1 - \cos \phi)^6 (1 + \cos \phi) \sin \phi d\phi d\theta;$$

$$\begin{bmatrix} u = 1 - \cos \phi \\ du = \sin \phi d\phi \end{bmatrix} \rightarrow \frac{1}{5} \int_0^{2\pi} \int_0^2 u^6 (2-u) du d\theta = \frac{1}{5} \int_0^{2\pi} \left[\frac{2u^7}{7} - \frac{u^8}{8} \right]_0^2 d\theta = \frac{1}{5} \int_0^{2\pi} \left(\frac{1}{7} - \frac{1}{8} \right) 2^8 d\theta$$

$$= \frac{1}{5} \int_0^{2\pi} \frac{2^3 \cdot 2^5}{56} d\theta = \frac{32}{35} \int_0^{2\pi} d\theta = \frac{64\pi}{35}$$

CHAPTER 12 ADDITIONAL EXERCISES—THEORY, EXAMPLES, APPLICATIONS

1. (a) $V = \int_{-3}^2 \int_x^{6-x^2} x^2 dy dx$

(b) $V = \int_{-3}^2 \int_x^{6-x^2} \int_0^x dz dy dx$

(c) $V = \int_{-3}^2 \int_x^{6-x^2} x^2 dy dx = \int_{-3}^2 (6x^2 - x^4 - x^3) dx = \left[2x^3 - \frac{x^5}{5} - \frac{x^4}{4} \right]_{-3}^2 = \frac{125}{4}$

2. Place the sphere's center at the origin with the surface of the water at $z = -3$. Then

$9 = 25 - x^2 - y^2 \Rightarrow x^2 + y^2 = 16$ is the projection of the volume of water onto the xy -plane

$$\Rightarrow V = \int_0^{2\pi} \int_0^4 \int_{-\sqrt{25-r^2}}^{-3} dz r dr d\theta = \int_0^{2\pi} \int_0^4 (r\sqrt{25-r^2} - 3r) dr d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(25-r^2)^{3/2} - \frac{3}{2}r^2 \right]_0^4 d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3}(9)^{3/2} - 24 + \frac{1}{3}(25)^{3/2} \right] d\theta = \int_0^{2\pi} \frac{26}{3} d\theta = \frac{52\pi}{3}$$

3. Using cylindrical coordinates, $V = \int_0^{2\pi} \int_0^1 \int_0^{2-r(\cos \theta + \sin \theta)} dz r dr d\theta = \int_0^{2\pi} \int_0^1 (2r - r^2 \cos \theta - r^2 \sin \theta) dr d\theta$

$$= \int_0^{2\pi} \left(1 - \frac{1}{3} \cos \theta - \frac{1}{3} \sin \theta \right) d\theta = \left[\theta + \frac{1}{3} \sin \theta - \frac{1}{3} \cos \theta \right]_0^{2\pi} = 2\pi$$

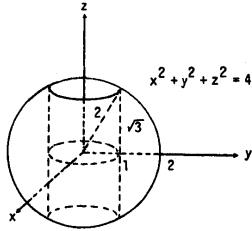
$$\begin{aligned} 4. V &= 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 (r\sqrt{2-r^2} - r^3) dr \, d\theta = 4 \int_0^{\pi/2} \left[-\frac{1}{3}(2-r^2)^{3/2} - \frac{r^4}{4} \right]_0^1 d\theta \\ &= 4 \int_0^{\pi/2} \left(-\frac{1}{3} - \frac{1}{4} + \frac{2\sqrt{2}}{3} \right) d\theta = \left(\frac{8\sqrt{2}-7}{3} \right) \int_0^{\pi/2} d\theta = \frac{\pi(8\sqrt{2}-7)}{6} \end{aligned}$$

5. The surfaces intersect when $3 - x^2 - y^2 = 2x^2 + 2y^2 \Rightarrow x^2 + y^2 = 1$. Thus the volume is

$$V = 4 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{2x^2+2y^2}^{3-x^2-y^2} dz \, dy \, dx = 4 \int_0^{\pi/2} \int_0^1 \int_{2r^2}^{3-r^2} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 (3r - 3r^3) dr \, d\theta = 3 \int_0^{\pi/2} d\theta = \frac{3\pi}{2}$$

$$\begin{aligned} 6. V &= 8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{2 \sin \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{64}{3} \int_0^{\pi/2} \int_0^{\pi/2} \sin^4 \phi \, d\phi \, d\theta \\ &= \frac{64}{3} \int_0^{\pi/2} \left[-\frac{\sin^3 \phi \cos \phi}{4} \Big|_0^{\pi/2} + \frac{3}{4} \int_0^{\pi/2} \sin^2 \phi \, d\phi \right] d\theta = 16 \int_0^{\pi/2} \left[\frac{\phi}{2} - \frac{\sin 2\phi}{4} \right]_0^{\pi/2} d\theta = 4\pi \int_0^{\pi/2} d\theta = 2\pi^2 \end{aligned}$$

7. (a) The radius of the hole is 1, and the radius of the sphere is 2.



$$(b) V = 2 \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-z^2}} r \, dr \, dz \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} (3 - z^2) dz \, d\theta = 2\sqrt{3} \int_0^{2\pi} d\theta = 4\sqrt{3}\pi$$

$$\begin{aligned} 8. V &= \int_0^{\pi} \int_0^{3 \sin \theta} \int_0^{\sqrt{9-r^2}} dz \, r \, dr \, d\theta = \int_0^{\pi} \int_0^{3 \sin \theta} r \sqrt{9-r^2} dr \, d\theta = \int_0^{\pi} \left[-\frac{1}{3}(9-r^2)^{3/2} \right]_0^{3 \sin \theta} d\theta \\ &= \int_0^{\pi} \left[-\frac{1}{3}(9-9 \sin^2 \theta)^{3/2} + \frac{1}{3}(9)^{3/2} \right] d\theta = 9 \int_0^{\pi} \left[1 - (1 - \sin^2 \theta)^{3/2} \right] d\theta = 9 \int_0^{\pi} (1 - \cos^3 \theta) d\theta \\ &= \int_0^{\pi} (1 - \cos \theta + \sin^2 \theta \cos \theta) d\theta = 9 \left[\theta + \sin \theta + \frac{\sin^3 \theta}{3} \right]_0^{\pi} = 9\pi \end{aligned}$$

9. The surfaces intersect when $x^2 + y^2 = \frac{x^2 + y^2 + 1}{2} \Rightarrow x^2 + y^2 = 1$. Thus the volume in cylindrical

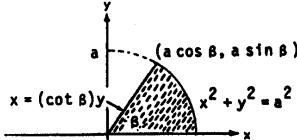
$$\text{coordinates is } V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{(r^2+1)/2} dz \ r \ dr \ d\theta = 4 \int_0^{\pi/2} \int_0^1 \left(\frac{r}{2} - \frac{r^3}{2} \right) dr \ d\theta = 4 \int_0^{\pi/2} \left[\frac{r^2}{4} - \frac{r^4}{8} \right]_0^1 d\theta \\ = \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{\pi}{4}$$

$$10. V = \int_0^{\pi/2} \int_1^2 \int_0^{r^2 \sin \theta \cos \theta} dz \ r \ dr \ d\theta = \int_0^{\pi/2} \int_1^2 r^3 \sin \theta \cos \theta \ dr \ d\theta = \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_1^2 \sin \theta \cos \theta \ d\theta \\ = \frac{15}{4} \int_0^{\pi/2} \sin \theta \cos \theta \ d\theta = \frac{15}{4} \left[\frac{\sin^2 \theta}{2} \right]_0^{\pi/2} = \frac{15}{8}$$

$$11. \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \int_0^\infty \int_a^b e^{-xy} dy dx = \int_a^b \int_0^\infty e^{-xy} dx dy = \int_a^b \left(\lim_{t \rightarrow \infty} \int_0^t e^{-xy} dx \right) dy \\ = \int_a^b \lim_{t \rightarrow \infty} \left[-\frac{e^{-xy}}{y} \right]_0^t dy = \int_a^b \lim_{t \rightarrow \infty} \left(\frac{1}{y} - \frac{e^{-yt}}{y} \right) dy = \int_a^b \frac{1}{y} dy = [\ln y]_a^b = \ln \left(\frac{b}{a} \right)$$

12. (a) The region of integration is sketched at the right

$$\Rightarrow \int_0^{a \sin \beta} \int_y^{a \sqrt{a^2 - y^2}} \ln(x^2 + y^2) dx dy = \int_0^\beta \int_0^a r \ln(r^2) dr d\theta; \\ \begin{bmatrix} u = r^2 \\ du = 2r dr \end{bmatrix} \rightarrow \frac{1}{2} \int_0^\beta \int_0^{a^2} \ln u \ du d\theta = \frac{1}{2} \int_0^\beta [u \ln u - u]_0^{a^2} d\theta \\ = \frac{1}{2} \int_0^\beta [2a^2 \ln a - a^2 - \lim_{t \rightarrow 0} t \ln t] d\theta = \frac{a^2}{2} \int_0^\beta (2 \ln a - 1) d\theta = a^2 \beta \left(\ln a - \frac{1}{2} \right)$$



$$(b) \int_0^{a \cos \beta} \int_0^{(\tan \beta)x} \ln(x^2 + y^2) dy dx + \int_{a \cos \beta}^a \int_0^{\sqrt{a^2 - x^2}} \ln(x^2 + y^2) dy dx$$

$$13. \int_0^x \int_0^u e^{m(x-t)} f(t) dt du = \int_0^x \int_t^x e^{m(x-t)} f(t) du dt = \int_0^x (x-t) e^{m(x-t)} f(t) dt; \text{ also}$$

$$\int_0^x \int_0^v \int_0^u e^{m(x-t)} f(t) dt du dv = \int_0^x \int_t^x \int_t^v e^{m(x-t)} f(t) du dv dt = \int_0^x (v-t) e^{m(x-t)} f(t) dv dt$$

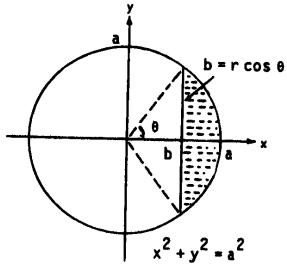
$$= \int_0^x \left[\frac{1}{2}(v-t)^2 e^{m(x-t)} f(t) \right]_t^x dt = \int_0^x \frac{(x-t)^2}{2} e^{m(x-t)} f(t) dt$$

$$\begin{aligned} 14. \quad & \int_0^1 f(x) \left(\int_0^x g(x-y) f(y) dy \right) dx = \int_0^1 \int_0^x g(x-y) f(x) f(y) dy dx \\ & = \int_0^1 \int_y^1 g(x-y) f(x) f(y) dx dy = \int_0^1 f(y) \left(\int_y^1 g(x-y) f(x) dx \right) dy; \\ & \int_0^1 \int_0^1 g(|x-y|) f(x) f(y) dx dy = \int_0^1 \int_0^x g(x-y) f(x) f(y) dy dx + \int_0^1 \int_x^1 g(y-x) f(x) f(y) dy dx \\ & = \int_0^1 \int_y^1 g(x-y) f(x) f(y) dx dy + \int_0^1 \int_0^x g(y-x) f(x) f(y) dy dx \\ & = \int_0^1 \int_y^1 g(x-y) f(x) f(y) dx dy + \underbrace{\int_0^1 \int_y^1 g(x-y) f(y) f(x) dx dy}_{\text{simply interchange } x \text{ and } y \text{ variable names}} \\ & = 2 \int_0^1 \int_y^1 g(x-y) f(x) f(y) dx dy, \text{ and the statement now follows.} \end{aligned}$$

$$\begin{aligned} 15. \quad I_o(a) &= \int_0^a \int_0^{x/a^2} (x^2 + y^2) dy dx = \int_0^a \left[x^2 y + \frac{y^3}{3} \right]_0^{x/a^2} dx = \int_0^a \left(\frac{x^3}{a^2} + \frac{x^3}{3a^6} \right) dx = \left[\frac{x^4}{4a^2} + \frac{x^4}{12a^6} \right]_0^a \\ &= \frac{a^2}{4} + \frac{1}{12} a^{-2}; I'_o(a) = \frac{1}{2} a - \frac{1}{6} a^{-3} = 0 \Rightarrow a^4 = \frac{1}{3} \Rightarrow a = \sqrt[4]{\frac{1}{3}} = \frac{1}{\sqrt[4]{3}}. \text{ Since } I''_o(a) = \frac{1}{2} + \frac{1}{2} a^{-4} > 0, \text{ the value of } a \text{ does provide a } \underline{\text{minimum}} \text{ for the polar moment of inertia } I_o(a). \end{aligned}$$

$$16. \quad I_o = \int_0^2 \int_{2x}^4 (x^2 + y^2)(3) dy dx = 3 \int_0^2 \left(4x^2 - \frac{14x^3}{3} + \frac{64}{3} \right) dx = 104$$

$$\begin{aligned}
 17. M &= \int_{-\theta}^{\theta} \int_0^a r dr d\theta = \int_{-\theta}^{\theta} \left(\frac{a^2}{2} - \frac{b^2}{2} \sec^2 \theta \right) d\theta \\
 &= a^2 \theta - b^2 \tan \theta = a^2 \cos^{-1} \left(\frac{b}{a} \right) - b^2 \left(\frac{\sqrt{a^2 - b^2}}{b} \right) \\
 &= a^2 \cos^{-1} \left(\frac{b}{a} \right) - b \sqrt{a^2 - b^2}; I_o = \int_{-\theta}^{\theta} \int_0^a r^3 dr d\theta
 \end{aligned}$$



$$\begin{aligned}
 &= \frac{1}{4} \int_{-\theta}^{\theta} (a^4 - b^4 \sec^4 \theta) d\theta = \frac{1}{4} \int_{-\theta}^{\theta} [a^4 - b^4(1 + \tan^2 \theta)(\sec^2 \theta)] d\theta = \frac{1}{4} \left[a^4 \theta - b^4 \tan \theta - \frac{b^4 \tan^3 \theta}{3} \right]_{-\theta}^{\theta} \\
 &= \frac{a^4 \theta}{2} - \frac{b^4 \tan \theta}{2} - \frac{b^4 \tan^3 \theta}{6} = \frac{1}{2} a^4 \cos^{-1} \left(\frac{b}{a} \right) - \frac{1}{2} b^3 \sqrt{a^2 - b^2} - \frac{1}{6} b (a^2 - b^2)^{3/2}
 \end{aligned}$$

$$\begin{aligned}
 18. M &= \int_{-2}^2 \int_{1-(y^2/4)}^{2-(y^2/2)} dx dy = \int_{-2}^2 \left(1 - \frac{y^2}{4} \right) dy = \left[y - \frac{y^3}{12} \right]_{-2}^2 = \frac{8}{3}; M_y = \int_{-2}^2 \int_{1-(y^2/4)}^{2-(y^2/2)} x dx dy \\
 &= \int_{-2}^2 \left[\frac{x^2}{2} \right]_{1-(y^2/4)}^{2-(y^2/2)} dy = \frac{3}{32} \int_{-2}^2 (16 - 8y^2 + y^4) dy = \frac{3}{16} \left[16y - \frac{8y^3}{3} + \frac{y^5}{5} \right]_0^2 \\
 &= \frac{3}{16} \left(32 - \frac{64}{3} + \frac{32}{5} \right) = \left(\frac{3}{16} \right) \left(\frac{32 \cdot 8}{15} \right) = \frac{48}{15} \Rightarrow \bar{x} = \frac{M_y}{M} = \left(\frac{48}{15} \right) \left(\frac{3}{8} \right) = \frac{6}{5}, \text{ and } \bar{y} = 0 \text{ by symmetry}
 \end{aligned}$$

$$\begin{aligned}
 19. \int_0^a \int_0^b e^{\max(b^2 x^2, a^2 y^2)} dy dx &= \int_0^a \int_0^{bx/a} e^{b^2 x^2} dy dx + \int_0^a \int_0^{ay/b} e^{a^2 y^2} dx dy \\
 &= \int_0^a \left(\frac{b}{a} x \right) e^{b^2 x^2} dx + \int_0^b \left(\frac{a}{b} y \right) e^{a^2 y^2} dy = \left[\frac{1}{2ab} e^{b^2 x^2} \right]_0^a + \left[\frac{1}{2ba} e^{a^2 y^2} \right]_0^b = \frac{1}{2ab} (e^{b^2 a^2} - 1) + \frac{1}{2ab} (e^{a^2 b^2} - 1) \\
 &= \frac{1}{ab} (e^{a^2 b^2} - 1)
 \end{aligned}$$

$$\begin{aligned}
 20. \int_{y_0}^{y_1} \int_{x_0}^{x_1} \frac{\partial^2 F(x, y)}{\partial x \partial y} dx dy &= \int_{y_0}^{y_1} \left[\frac{\partial F(x, y)}{\partial y} \right]_{x_0}^{x_1} dy = \int_{y_0}^{y_1} \left[\frac{\partial F(x_1, y)}{\partial y} - \frac{\partial F(x_0, y)}{\partial y} \right] dy = [F(x_1, y) - F(x_0, y)]_{y_0}^{y_1} \\
 &= F(x_1, y_1) - F(x_0, y_1) - F(x_1, y_0) + F(x_0, y_0)
 \end{aligned}$$

21. (a) (i) Fubini's Theorem
(ii) Treating G(y) as a constant
(iii) Algebraic rearrangement
(iv) The definite integral is a constant number

$$(b) \int_0^{\ln 2} \int_0^{\pi/2} e^x \cos y \, dy \, dx = \left(\int_0^{\ln 2} e^x \, dx \right) \left(\int_0^{\pi/2} \cos y \, dy \right) = (e^{\ln 2} - e^0) \left(\sin \frac{\pi}{2} - \sin 0 \right) = (1)(1) = 1$$

$$(c) \int_1^2 \int_{-1}^1 \frac{x}{y^2} \, dx \, dy = \left(\int_1^2 \frac{1}{y^2} \, dy \right) \left(\int_{-1}^1 x \, dx \right) = \left[-\frac{1}{y} \right]_1^2 \left[\frac{x^2}{2} \right]_{-1}^1 = \left(-\frac{1}{2} + 1 \right) \left(\frac{1}{2} - \frac{1}{2} \right) = 0$$

22. (a) $\nabla f = u_1 i + u_2 j \Rightarrow D_u f = u_1 x + u_2 y$; the area of the region of integration is $\frac{1}{2}$

$$\Rightarrow \text{average} = 2 \int_0^1 \int_0^{1-x} (u_1 x + u_2 y) \, dy \, dx = 2 \int_0^1 \left[u_1 x (1-x) + \frac{1}{2} u_2 (1-x)^2 \right] dx \\ = 2 \left[u_1 \left(\frac{x^2}{2} - \frac{x^3}{3} \right) - \left(\frac{1}{2} u_2 \right) \frac{(1-x)^3}{3} \right]_0^1 = 2 \left(\frac{1}{6} u_1 + \frac{1}{6} u_2 \right) = \frac{1}{3} (u_1 + u_2)$$

$$(b) \text{average} = \frac{1}{\text{area}} \int_R \int (u_1 x + u_2 y) \, dA = \frac{u_1}{\text{area}} \int_R \int x \, dA + \frac{u_2}{\text{area}} \int_R \int y \, dA = u_1 \left(\frac{M_y}{M} \right) + u_2 \left(\frac{M_x}{M} \right) = u_1 \bar{x} + u_2 \bar{y}$$

$$23. (a) I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy = \int_0^{\pi/2} \int_0^\infty \left(e^{-r^2} \right) r \, dr \, d\theta = \int_0^{\pi/2} \left[\lim_{b \rightarrow \infty} \int_0^b r e^{-r^2} \, dr \right] d\theta \\ = -\frac{1}{2} \int_0^{\pi/2} \lim_{b \rightarrow \infty} (e^{-b^2} - 1) \, d\theta = \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}$$

$$(b) \Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t} \, dt = \int_0^\infty (y^2)^{-1/2} e^{-y^2} (2y) \, dy = 2 \int_0^\infty e^{-y^2} \, dy = 2 \left(\frac{\sqrt{\pi}}{2} \right) = \sqrt{\pi}, \text{ where } y = \sqrt{t}$$

$$24. Q = \int_0^{2\pi} \int_0^R kr^2(1-\sin \theta) \, dr \, d\theta = \frac{kR^3}{3} \int_0^{2\pi} (1-\sin \theta) \, d\theta = \frac{kR^3}{3} [\theta + \cos \theta]_0^{2\pi} = \frac{2\pi k R^3}{3}$$

$$25. \text{For a height } h \text{ in the bowl the volume of water is } V = \int_{-\sqrt{h}}^{\sqrt{h}} \int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}} \int_0^h dz \, dy \, dx$$

$$= \int_{-\sqrt{h}}^{\sqrt{h}} \int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}} (h - x^2 - y^2) \, dy \, dx = \int_0^{2\pi} \int_0^{\sqrt{h}} (h - r^2) r \, dr \, d\theta = \int_0^{2\pi} \left[\frac{hr^2}{2} - \frac{r^4}{4} \right]_0^{\sqrt{h}} d\theta = \int_0^{2\pi} \frac{h^2}{4} d\theta = \frac{h^2 \pi}{2}.$$

Since the top of the bowl has area 10π , then we calibrate the bowl by comparing it to a right circular cylinder whose cross sectional area is 10π from $z = 0$ to $z = 10$. If such a cylinder contains $\frac{h^2 \pi}{2}$ cubic inches of water to a depth w then we have $10\pi w = \frac{h^2 \pi}{2} \Rightarrow w = \frac{h^2}{20}$. So for 1 inch of rain, $w = 1$ and $h = \sqrt{20}$; for 3 inches of rain, $w = 3$ and $h = \sqrt{60}$.

26. (a) An equation for the satellite dish in standard position

is $z = \frac{1}{2}x^2 + \frac{1}{2}y^2$. Since the axis is tilted 30° , a unit vector $\mathbf{v} = 0\mathbf{i} + a\mathbf{j} + b\mathbf{k}$ normal to the plane of the

water level satisfies $b = \mathbf{v} \cdot \mathbf{k} = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

$$\Rightarrow a = -\sqrt{1 - b^2} = -\frac{1}{2} \Rightarrow \mathbf{v} = -\frac{1}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}$$

$$\Rightarrow -\frac{1}{2}(y-1) + \frac{\sqrt{3}}{2}(z-\frac{1}{2}) = 0 \Rightarrow z = \frac{1}{\sqrt{3}}y + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)$$

is an equation of the plane of the water level. Therefore

$$\text{the volume of water is } V = \int_R \int_{\frac{1}{2}x^2 + \frac{1}{2}y^2}^{\frac{1}{\sqrt{3}}y + \frac{1}{2} - \frac{1}{\sqrt{3}}} dz dy dx, \text{ where } R \text{ is the interior of the ellipse}$$

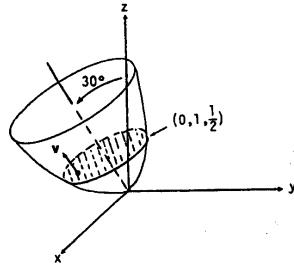
$$x^2 + y^2 - \frac{2}{3}y - 1 + \frac{2}{\sqrt{3}} = 0. \text{ When } x = 0, \text{ then } y = \alpha \text{ or } y = \beta, \text{ where } \alpha = \frac{\frac{2}{3} + \sqrt{\frac{4}{9} - 4\left(\frac{2}{\sqrt{3}} - 1\right)}}{2}$$

$$\text{and } \beta = \frac{\frac{2}{3} - \sqrt{\frac{4}{9} - 4\left(\frac{2}{\sqrt{3}} - 1\right)}}{2} \Rightarrow V = \int_{\alpha}^{\beta} \int_{-\left(\frac{2}{3}y + 1 - \frac{2}{\sqrt{3}}y^2\right)^{1/2}}^{\left(\frac{2}{3}y + 1 - \frac{2}{\sqrt{3}}y^2\right)^{1/2}} \int_{\frac{1}{2}x^2 + \frac{1}{2}y^2}^{\frac{1}{\sqrt{3}}y + \frac{1}{2} - \frac{1}{\sqrt{3}}} dz dx dy$$

- (b) $x = 0 \Rightarrow z = \frac{1}{2}y^2$ and $\frac{dz}{dy} = y$; $y = 1 \Rightarrow \frac{dz}{dy} = 1 \Rightarrow$ the tangent line has slope 1 or a 45° slant
 \Rightarrow at 45° and thereafter, the dish will not hold water.

27. The cylinder is given by $x^2 + y^2 = 1$ from $z = 1$ to $\infty \Rightarrow \int_D \int z(r^2 + z^2)^{-5/2} dV$

$$\begin{aligned} &= \int_0^{2\pi} \int_0^1 \int_1^\infty \frac{z}{(r^2 + z^2)^{5/2}} dz r dr d\theta = \lim_{a \rightarrow \infty} \int_0^{2\pi} \int_0^1 \int_1^a \frac{rz}{(r^2 + z^2)^{5/2}} dz dr d\theta \\ &= \lim_{a \rightarrow \infty} \int_0^{2\pi} \int_0^1 \left[\left(-\frac{1}{3} \right) \frac{r}{(r^2 + z^2)^{3/2}} \right]_1^a dr d\theta = \lim_{a \rightarrow \infty} \int_0^{2\pi} \int_0^1 \left[\left(-\frac{1}{3} \right) \frac{r}{(r^2 + a^2)^{3/2}} + \left(\frac{1}{3} \right) \frac{r}{(r^2 + 1)^{3/2}} \right] dr d\theta \\ &= \lim_{a \rightarrow \infty} \int_0^{2\pi} \left[\frac{1}{3} (r^2 + a^2)^{-1/2} - \frac{1}{3} (r^2 + 1)^{-1/2} \right]_0^1 dr = \lim_{a \rightarrow \infty} \int_0^{2\pi} \left[\frac{1}{3} (1 + a^2)^{-1/2} - \frac{1}{3} (2^{-1/2}) - \frac{1}{3} (a^2)^{-1/2} + \frac{1}{3} \right] dr \\ &= \lim_{a \rightarrow \infty} 2\pi \left[\frac{1}{3} (1 + a^2)^{-1/2} - \frac{1}{3} \left(\frac{\sqrt{2}}{2} \right) - \frac{1}{3} \left(\frac{1}{a} \right) + \frac{1}{3} \right] = 2\pi \left[\frac{1}{3} - \left(\frac{1}{3} \right) \frac{\sqrt{2}}{2} \right]. \end{aligned}$$



28. Let's see?

$$\text{The length of the "unit" line segment is: } L = 2 \int_0^1 dx = 2$$

$$\text{The area of the unit circle is: } A = 4 \int_0^1 \int_0^{\sqrt{1-x^2}} dy dx = \pi.$$

$$\text{The volume of the unit sphere is: } V = 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz dy dx = \frac{4}{3}\pi.$$

Therefore, the hypervolume of the unit 4-sphere should be:

$$V_{\text{hyper}} = 16 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \int_0^{\sqrt{1-x^2-y^2-z^2}} dw dz dy dx.$$

Mathematica is able to handle this integral, but we'll use the brute force approach.

$$\begin{aligned} V_{\text{hyper}} &= 16 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \int_0^{\sqrt{1-x^2-y^2-z^2}} dw dz dy dx = 16 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \int_0^{\sqrt{1-x^2-y^2-z^2}} dz dy dx \\ &= 16 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \sqrt{1-x^2-y^2} \sqrt{1-\frac{z^2}{1-x^2-y^2}} dz dy dx = \left[\begin{array}{l} \frac{z}{\sqrt{1-x^2-y^2}} = \cos \theta \\ dz = -\sqrt{1-x^2-y^2} \sin \theta d\theta \end{array} \right] \\ &= 16 \int_0^1 \int_0^{\sqrt{1-x^2}} (1-x^2-y^2) \int_{\pi/2}^0 -\sqrt{1-\cos^2 \theta} \sin \theta d\theta dy dx = 16 \int_0^1 \int_0^{\sqrt{1-x^2}} (1-x^2-y^2) \int_{\pi/2}^0 -\sin^2 \theta d\theta dy dx \\ &= 16 \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{4}(1-x^2-y^2) dy dx = 4\pi \int_0^1 \sqrt{1-x^2} - x^2 \sqrt{1-x^2} - \frac{1}{3}(1-x^2)^{3/2} dx \\ &= 4\pi \int_0^1 \sqrt{1-x^2} \left[(1-x^2) - \frac{1-x^2}{3} \right] dx = \frac{8}{3}\pi \int_0^1 (1-x^2)^{3/2} dx = \left[\begin{array}{l} x = \cos \theta \\ dx = -\sin \theta d\theta \end{array} \right] = -\frac{8}{3}\pi \int_{\pi/2}^0 \sin^4 \theta d\theta \\ &= -\frac{8}{3}\pi \int_{\pi/2}^0 \left[\frac{1-\cos 2\theta}{2} \right]^2 = -\frac{2}{3}\pi \int_{\pi/2}^0 (1-2\cos 2\theta + \cos^2 2\theta) d\theta = -\frac{2}{3}\pi \int_{\pi/2}^0 \left(\frac{3}{2} - 2\cos 2\theta + \frac{\cos 4\theta}{2} \right) d\theta = \frac{\pi^2}{2} \end{aligned}$$

NOTES: