

$$22. \text{ (a) } \mathbf{r} = (\cos 4t)\mathbf{i} + (\sin 4t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = (-4 \sin 4t)\mathbf{i} + (4 \cos 4t)\mathbf{j} + 4\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-4 \sin 4t)^2 + (4 \cos 4t)^2 + 4^2} \\ = \sqrt{32} = 4\sqrt{2} \Rightarrow \text{Length} = \int_0^{\pi/2} 4\sqrt{2} \, dt = [4\sqrt{2}t]_0^{\pi/2} = 2\pi\sqrt{2}$$

$$\text{ (b) } \mathbf{r} = \left(\cos \frac{t}{2}\right)\mathbf{i} + \left(\sin \frac{t}{2}\right)\mathbf{j} + \frac{t}{2}\mathbf{k} \Rightarrow \mathbf{v} = \left(-\frac{1}{2} \sin \frac{t}{2}\right)\mathbf{i} + \left(\frac{1}{2} \cos \frac{t}{2}\right)\mathbf{j} + \frac{1}{2}\mathbf{k} \\ \Rightarrow |\mathbf{v}| = \sqrt{\left(-\frac{1}{2} \sin \frac{t}{2}\right)^2 + \left(\frac{1}{2} \cos \frac{t}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{2}}{2} \Rightarrow \text{Length} = \int_0^{4\pi} \frac{\sqrt{2}}{2} \, dt = \left[\frac{\sqrt{2}}{2}t\right]_0^{4\pi} = 2\pi\sqrt{2}$$

$$\text{ (c) } \mathbf{r} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} - t\mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j} - \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-\sin t)^2 + (-\cos t)^2 + (-1)^2} = \sqrt{1+1} \\ = \sqrt{2} \Rightarrow \text{Length} = \int_{-2\pi}^0 \sqrt{2} \, dt = [\sqrt{2}t]_{-2\pi}^0 = 2\pi\sqrt{2}$$

$$23. \mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + (\cos t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1 + \cos^2 t} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (1 + \cos^2 t)^{-1/2} \mathbf{i} + \cos t (1 + \cos^2 t)^{-1/2} \mathbf{j}$$

$$\frac{d\mathbf{T}}{dt} = \sin t \cos t (1 + \cos^2 t)^{-3/2} \mathbf{i} + \left[\sin t \cos^2 t (1 + \cos^2 t)^{-3/2} - \sin t (1 + \cos^2 t)^{-1/2} \right] \mathbf{j}$$

$$\kappa(0) = \frac{1}{|\mathbf{v}(\frac{\pi}{2})|} \left| \frac{d\mathbf{T}}{dt} \left(\frac{\pi}{2} \right) \right| = \frac{1}{\sqrt{1}} |0\mathbf{i} - \mathbf{j}| = (1)\sqrt{0^2 + 1^2} = 1 \Rightarrow \rho(0) = \frac{1}{\kappa(0)} = 1$$

Since $\frac{d\mathbf{T}}{dt}(0) = -\mathbf{j}$, the curve is concave down at $(\frac{\pi}{2}, 1)$ and the center of the circle of curvature is at

$$\left(\frac{\pi}{2}, 0\right) \Rightarrow \left(x - \frac{\pi}{2}\right)^2 + (y - 0)^2 = 1 \Rightarrow \left(x - \frac{\pi}{2}\right)^2 + y^2 = 1 \text{ is an equation of the circle of curvature.}$$

$$24. \mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(\frac{2}{t}\right)\mathbf{i} - \left(1 - \frac{1}{t^2}\right)\mathbf{j} = \left(\frac{2}{t}\right)\mathbf{i} + \left(\frac{1-t^2}{t^2}\right)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{\left(\frac{2}{t}\right)^2 + \left(\frac{1-t^2}{t^2}\right)^2} = \frac{1+t^2}{t^2}$$

$$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{2t}{1+t^2}\right)\mathbf{i} + \left(\frac{1-t^2}{1+t^2}\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(\frac{2-2t^2}{(1+t^2)^2}\right)\mathbf{i} - \left(\frac{4t}{(1+t^2)^2}\right)\mathbf{j}$$

$$\kappa(0) = \frac{1}{|\mathbf{v}(1)|} \left| \frac{d\mathbf{T}}{dt}(1) \right| = \frac{1}{2} |0\mathbf{i} - \mathbf{j}| = \left(\frac{1}{2}\right)\sqrt{(0)^2 + 1^2} = \frac{1}{2} \Rightarrow \rho(0) = \frac{1}{\kappa(0)} = 2$$

Since $\frac{d\mathbf{T}}{dt}(1) = -\mathbf{j}$, the curve is concave down at $(0, -2)$ and the center of the circle of curvature is at

$$(0, -4) \Rightarrow x^2 + (y + 4)^2 = 4 \text{ is an equation of the circle of curvature.}$$

10.7 THE TNB FRAME: TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION

$$1. \mathbf{r} = (3 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = (3 \cos t)\mathbf{i} + (-3 \sin t)\mathbf{j} + 4\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(3 \cos t)^2 + (-3 \sin t)^2 + 4^2}$$

$$= \sqrt{25} = 5 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{3}{5} \cos t\right)\mathbf{i} - \left(\frac{3}{5} \sin t\right)\mathbf{j} + \frac{4}{5}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{3}{5} \sin t\right)\mathbf{i} - \left(\frac{3}{5} \cos t\right)\mathbf{j}$$

$$\Rightarrow \left| \frac{dT}{dt} \right| = \sqrt{\left(-\frac{3}{5} \sin t\right)^2 + \left(-\frac{3}{5} \cos t\right)^2} = \frac{3}{5} \Rightarrow N = \frac{\left(\frac{dT}{dt}\right)}{\left|\frac{dT}{dt}\right|} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}; \mathbf{a} = (-3 \sin t)\mathbf{i} + (-3 \cos t)\mathbf{j}$$

$$\Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 \cos t & -3 \sin t & 4 \\ -3 \sin t & -3 \cos t & 0 \end{vmatrix} = (12 \cos t)\mathbf{i} - (12 \sin t)\mathbf{j} - 9\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}|$$

$$= \sqrt{(12 \cos t)^2 + (-12 \sin t)^2 + (-9)^2} = \sqrt{225} = 15 \Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{15}{5^3} = \frac{3}{25}; \mathbf{B} = \mathbf{T} \times \mathbf{N}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{5} \cos t & -\frac{3}{5} \sin t & \frac{4}{5} \\ -\sin t & -\cos t & 0 \end{vmatrix} = \left(\frac{4}{5} \cos t\right)\mathbf{i} - \left(\frac{4}{5} \sin t\right)\mathbf{j} - \frac{3}{5}\mathbf{k}; \frac{d\mathbf{a}}{dt} = (-3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j}$$

$$\Rightarrow \tau = \frac{\begin{vmatrix} 3 \cos t & -3 \sin t & 4 \\ -3 \sin t & -3 \cos t & 0 \\ -3 \cos t & 3 \sin t & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{-36 \sin^2 t - 36 \cos^2 t}{15^2} = -\frac{4}{25}$$

$$2. \mathbf{r} = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j} + 3\mathbf{k} \Rightarrow \mathbf{v} = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t \cos t)^2 + (t \sin t)^2} = \sqrt{t^2} \\ = |\mathbf{t}| = t, \text{ if } t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}, t > 0 \Rightarrow \frac{dT}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$$

$$\Rightarrow \left| \frac{dT}{dt} \right| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{dT}{dt}\right)}{\left|\frac{dT}{dt}\right|} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}; \mathbf{a} = (\cos t - t \sin t)\mathbf{i} + (\sin t + t \cos t)\mathbf{j}$$

$$\Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t \cos t & t \sin t & 0 \\ \cos t - t \sin t & \sin t + t \cos t & 0 \end{vmatrix}$$

$$= [(t \cos t)(\sin t + t \cos t) - (t \sin t)(\cos t - t \sin t)]\mathbf{k} = t^2\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = \sqrt{(t^2)^2} = t^2$$

$$\Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{t^2}{t^3} = \frac{1}{t}; \mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \end{vmatrix} = (\cos^2 t + \sin^2 t)\mathbf{k} = \mathbf{k};$$

$$\frac{d\mathbf{a}}{dt} = (-2 \sin t - t \cos t)\mathbf{i} + (2 \cos t - t \sin t)\mathbf{j} \Rightarrow \tau = \frac{\begin{vmatrix} t \cos t & t \sin t & 0 \\ \cos t - t \sin t & \sin t + t \cos t & 0 \\ -2 \sin t - t \cos t & 2 \cos t - t \sin t & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2}$$

$$= \frac{0}{|\mathbf{v} \times \mathbf{a}|^2} = 0$$

$$3. \mathbf{r} = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + 2t\mathbf{k} \Rightarrow \mathbf{v} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} \Rightarrow$$

$$|\mathbf{v}| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2} = \sqrt{2e^{2t}} = e^t \sqrt{2};$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\cos t - \sin t}{\sqrt{2}} \right) \mathbf{i} + \left(\frac{\sin t + \cos t}{\sqrt{2}} \right) \mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(\frac{-\sin t - \cos t}{\sqrt{2}} \right) \mathbf{i} + \left(\frac{\cos t - \sin t}{\sqrt{2}} \right) \mathbf{j}$$

$$\Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{\left(\frac{-\sin t - \cos t}{\sqrt{2}} \right)^2 + \left(\frac{\cos t - \sin t}{\sqrt{2}} \right)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt} \right)}{\left| \frac{d\mathbf{T}}{dt} \right|} = \left(\frac{-\cos t - \sin t}{\sqrt{2}} \right) \mathbf{i} + \left(\frac{-\sin t + \cos t}{\sqrt{2}} \right) \mathbf{j};$$

$$\mathbf{a} = (-2e^t \sin t)\mathbf{i} + (2e^t \cos t)\mathbf{j} \Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t \cos t - e^t \sin t & e^t \sin t + e^t \cos t & 0 \\ -2e^t \sin t & 2e^t \cos t & 0 \end{vmatrix} = 2e^{2t} \mathbf{k}$$

$$\Rightarrow |\mathbf{v} \times \mathbf{a}| = \sqrt{(2e^{2t})^2} = 2e^{2t} \Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{2e^{2t}}{(e^t \sqrt{2})^3} = \frac{1}{e^t \sqrt{2}};$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\cos t - \sin t}{\sqrt{2}} & \frac{\sin t + \cos t}{\sqrt{2}} & 0 \\ \frac{-\cos t - \sin t}{\sqrt{2}} & \frac{-\sin t + \cos t}{\sqrt{2}} & 0 \end{vmatrix}$$

$$= \left[\frac{1}{2} (\cos t - \sin t)(-\sin t + \cos t) - \frac{1}{2} (-\cos t - \sin t)(\sin t + \cos t) \right] \mathbf{k}$$

$$= \left[\frac{1}{2} (\cos^2 t - 2 \cos t \sin t + \sin^2 t) + \frac{1}{2} (\cos^2 t + 2 \sin t \cos t + \sin^2 t) \right] \mathbf{k} = \mathbf{k};$$

$$\frac{d\mathbf{a}}{dt} = (-2e^t \sin t - 2e^t \cos t)\mathbf{i} + (2e^t \cos t - 2e^t \sin t)\mathbf{j}$$

$$\Rightarrow \tau = \frac{\begin{vmatrix} e^t \cos t - e^t \sin t & e^t \sin t + e^t \cos t & 0 \\ -2e^t \sin t & 2e^t \cos t & 0 \\ -2e^t \sin t - 2e^t \cos t & 2e^t \cos t - 2e^t \sin t & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = 0$$

$$4. \mathbf{r} = (6 \sin 2t)\mathbf{i} + (6 \cos 2t)\mathbf{j} + 5t\mathbf{k} \Rightarrow \mathbf{v} = (12 \cos 2t)\mathbf{i} - (12 \sin 2t)\mathbf{j} + 5\mathbf{k}$$

$$\Rightarrow |\mathbf{v}| = \sqrt{(12 \cos 2t)^2 + (-12 \sin 2t)^2 + 5^2} = \sqrt{169} = 13 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

$$= \left(\frac{12}{13} \cos 2t \right) \mathbf{i} - \left(\frac{12}{13} \sin 2t \right) \mathbf{j} + \frac{5}{13} \mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{24}{13} \sin 2t \right) \mathbf{i} - \left(\frac{24}{13} \cos 2t \right) \mathbf{j}$$

$$\Rightarrow \left| \frac{dT}{dt} \right| = \sqrt{\left(-\frac{24}{13} \sin 2t \right)^2 + \left(-\frac{24}{13} \cos 2t \right)^2} = \frac{24}{13} \Rightarrow \mathbf{N} = \frac{\left(\frac{dT}{dt} \right)}{\left| \frac{dT}{dt} \right|} = (-\sin 2t)\mathbf{i} - (\cos 2t)\mathbf{j};$$

$$\mathbf{a} = (-24 \sin 2t)\mathbf{i} - (24 \cos 2t)\mathbf{j} \Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 \cos 2t & -12 \sin 2t & 5 \\ -24 \sin 2t & -24 \cos 2t & 0 \end{vmatrix}$$

$$= (120 \cos 2t)\mathbf{i} - (120 \sin 2t)\mathbf{j} - 288\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = \sqrt{(120 \cos 2t)^2 + (-120 \sin 2t)^2 + (-288)^2} = 312$$

$$\Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{312}{13^3} = \frac{24}{169}; \mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{12}{13} \cos 2t & -\frac{12}{13} \sin 2t & \frac{5}{13} \\ -\sin 2t & -\cos 2t & 0 \end{vmatrix}$$

$$= \left(\frac{5}{13} \cos 2t \right)\mathbf{i} - \left(\frac{5}{13} \sin 2t \right)\mathbf{j} - \frac{12}{13}\mathbf{k}; \frac{d\mathbf{a}}{dt} = (-48 \cos 2t)\mathbf{i} + (48 \sin 2t)\mathbf{j}$$

$$\Rightarrow \tau = \frac{\begin{vmatrix} 12 \cos 2t & -12 \sin 2t & 5 \\ -24 \sin 2t & -24 \cos 2t & 0 \\ -48 \cos 2t & 48 \sin 2t & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = -\frac{(5)(24)(48)}{(312)^2} = -\frac{5 \cdot 1 \cdot 2}{13 \cdot 13} = -\frac{10}{169}$$

$$5. \mathbf{r} = \left(\frac{t^3}{3} \right)\mathbf{i} + \left(\frac{t^2}{2} \right)\mathbf{j}, t > 0 \Rightarrow \mathbf{v} = t^2\mathbf{i} + t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{t^4 + t^2} = t\sqrt{t^2 + 1}, \text{ since } t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

$$= \frac{t}{\sqrt{t^2 + 1}}\mathbf{i} + \frac{1}{\sqrt{t^2 + 1}}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \frac{1}{(t^2 + 1)^{3/2}}\mathbf{i} - \frac{t}{(t^2 + 1)^{3/2}}\mathbf{j} \Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{\left(\frac{1}{(t^2 + 1)^{3/2}} \right)^2 + \left(\frac{-t}{(t^2 + 1)^{3/2}} \right)^2}$$

$$= \sqrt{\frac{1 + t^2}{(t^2 + 1)^3}} = \frac{1}{t^2 + 1} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt} \right)}{\left| \frac{d\mathbf{T}}{dt} \right|} = \frac{1}{\sqrt{t^2 + 1}}\mathbf{i} - \frac{t}{\sqrt{t^2 + 1}}\mathbf{j}; \mathbf{a} = 2t\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2 & t & 0 \\ 2t & 1 & 0 \end{vmatrix} = -t^2\mathbf{k}$$

$$\Rightarrow |\mathbf{v} \times \mathbf{a}| = \sqrt{(-t^2)^2} = t^2 \Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{t^2}{(t\sqrt{t^2 + 1})^3} = \frac{1}{t(t^2 + 1)^{3/2}};$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{t}{\sqrt{t^2 + 1}} & \frac{1}{\sqrt{t^2 + 1}} & 0 \\ \frac{1}{\sqrt{t^2 + 1}} & \frac{-t}{\sqrt{t^2 + 1}} & 0 \end{vmatrix} = -\mathbf{k}; \frac{d\mathbf{a}}{dt} = 2\mathbf{i} \Rightarrow \tau = \frac{\begin{vmatrix} t^2 & t & 0 \\ 2t & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = 0$$

$$\begin{aligned}
6. \quad \mathbf{r} &= (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}, \quad 0 < t < \frac{\pi}{2} \Rightarrow \mathbf{v} = (-3 \cos^2 t \sin t)\mathbf{i} + (3 \sin^2 t \cos t)\mathbf{j} \\
\Rightarrow |\mathbf{v}| &= \sqrt{(-3 \cos^2 t \sin t)^2 + (3 \sin^2 t \cos t)^2} = \sqrt{9 \cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t} = 3 \cos t \sin t, \text{ since } 0 < t < \frac{\pi}{2} \\
\Rightarrow \mathbf{T} &= \frac{\mathbf{v}}{|\mathbf{v}|} = (-\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{\sin^2 t + \cos^2 t} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt} \right)}{\left| \frac{d\mathbf{T}}{dt} \right|} \\
&= (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{a} = (6 \cos t \sin^2 t - 3 \cos^3 t)\mathbf{i} + (6 \sin t \cos^2 t - 3 \sin^3 t)\mathbf{j} \\
\Rightarrow \mathbf{v} \times \mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 \cos^2 t \sin t & 3 \sin^2 t \cos t & 0 \\ 6 \cos t \sin^2 t - 3 \cos^3 t & 6 \sin t \cos^2 t - 3 \sin^3 t & 0 \end{vmatrix} \\
&= (-18 \sin^2 t \cos^4 t + 9 \cos^2 t \sin^4 t - 18 \sin^4 t \cos^2 t + 9 \sin^2 t \cos^4 t)\mathbf{k} = (-9 \sin^2 t \cos^4 t - 9 \cos^2 t \sin^4 t)\mathbf{k} \\
&= (-9 \sin^2 t \cos^2 t)(\cos^2 t + \sin^2 t)\mathbf{k} = (-9 \sin^2 t \cos^2 t)\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = 9 \sin^2 t \cos^2 t \Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} \\
&= \frac{9 \cos^2 t \sin^2 t}{(3 \cos t \sin t)^3} = \frac{1}{3 \cos t \sin t}; \quad \mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\cos t & \sin t & 0 \\ \sin t & \cos t & 0 \end{vmatrix} = -\mathbf{k}; \quad \frac{d\mathbf{a}}{dt} = f(t)\mathbf{i} + g(t)\mathbf{j} \text{ where} \\
f(t) &= \frac{d}{dt}(6 \cos t \sin^2 t - 3 \cos^3 t) \text{ and } g(t) = \frac{d}{dt}(6 \sin t \cos^2 t - 3 \sin^3 t) \\
\Rightarrow \tau &= \frac{\begin{vmatrix} -3 \cos^2 t \sin t & 3 \sin^2 t \cos t & 0 \\ 6 \cos t \sin^2 t - 3 \cos^3 t & 6 \sin t \cos^2 t - 3 \sin^3 t & 0 \\ f(t) & g(t) & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = 0
\end{aligned}$$

$$\begin{aligned}
7. \quad \mathbf{r} &= t\mathbf{i} + \left(a \cosh \frac{t}{a}\right)\mathbf{j}, \quad a > 0 \Rightarrow \mathbf{v} = \mathbf{i} + \left(\sinh \frac{t}{a}\right)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1 + \sinh^2 \left(\frac{t}{a}\right)} = \sqrt{\cosh^2 \left(\frac{t}{a}\right)} = \cosh \frac{t}{a} \\
\Rightarrow \mathbf{T} &= \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\operatorname{sech} \frac{t}{a}\right)\mathbf{i} + \left(\tanh \frac{t}{a}\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{1}{a} \operatorname{sech} \frac{t}{a} \tanh \frac{t}{a}\right)\mathbf{i} + \left(\frac{1}{a} \operatorname{sech}^2 \frac{t}{a}\right)\mathbf{j} \\
\Rightarrow \left| \frac{d\mathbf{T}}{dt} \right| &= \sqrt{\frac{1}{a^2} \operatorname{sech}^2 \left(\frac{t}{a}\right) \tanh^2 \left(\frac{t}{a}\right) + \frac{1}{a^2} \operatorname{sech}^4 \left(\frac{t}{a}\right)} = \frac{1}{a} \operatorname{sech} \left(\frac{t}{a}\right) \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt} \right)}{\left| \frac{d\mathbf{T}}{dt} \right|} = \left(-\tanh \frac{t}{a}\right)\mathbf{i} + \left(\operatorname{sech} \frac{t}{a}\right)\mathbf{j}; \\
\mathbf{a} &= \left(\frac{1}{a} \cosh \frac{t}{a}\right)\mathbf{j} \Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \sinh \left(\frac{t}{a}\right) & 0 \\ 0 & \frac{1}{a} \cosh \left(\frac{t}{a}\right) & 0 \end{vmatrix} = \left(\frac{1}{a} \cosh \frac{t}{a}\right)\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = \frac{1}{a} \cosh \left(\frac{t}{a}\right) \Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}
\end{aligned}$$

$$= \frac{\frac{1}{a} \cosh\left(\frac{t}{a}\right)}{\cosh^3\left(\frac{t}{a}\right)} = \frac{1}{a} \operatorname{sech}^2\left(\frac{t}{a}\right); \mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \operatorname{sech}\left(\frac{t}{a}\right) & \tanh\left(\frac{t}{a}\right) & 0 \\ -\tanh\left(\frac{t}{a}\right) & \operatorname{sech}\left(\frac{t}{a}\right) & 0 \end{vmatrix} = \mathbf{k}; \frac{d\mathbf{a}}{dt} = \frac{1}{a^2} \sinh\left(\frac{t}{a}\right) \mathbf{j}$$

$$\tau = \frac{\begin{vmatrix} 1 & \sinh\left(\frac{t}{a}\right) & 0 \\ 0 & \frac{1}{a} \cosh\left(\frac{t}{a}\right) & 0 \\ 0 & \frac{1}{a^2} \sinh\left(\frac{t}{a}\right) & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = 0$$

$$8. \mathbf{r} = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\sinh^2 t + (-\cosh t)^2 + 1} = \sqrt{2} \cosh t$$

$$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}} \tanh t\right)\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}} \operatorname{sech} t\right)\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(\frac{1}{\sqrt{2}} \operatorname{sech}^2 t\right)\mathbf{i} - \left(\frac{1}{\sqrt{2}} \operatorname{sech} t \tanh t\right)\mathbf{k}$$

$$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{1}{2} \operatorname{sech}^4 t + \frac{1}{2} \operatorname{sech}^2 t \tanh^2 t} = \frac{1}{\sqrt{2}} \operatorname{sech} t \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (\operatorname{sech} t)\mathbf{i} - (\tanh t)\mathbf{k};$$

$$\mathbf{a} = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} \Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sinh t & -\cosh t & 1 \\ \cosh t & -\sinh t & 0 \end{vmatrix} = (\sinh t)\mathbf{i} + (\cosh t)\mathbf{j} - \mathbf{k}$$

$$\Rightarrow |\mathbf{v} \times \mathbf{a}| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{2} \cosh t \Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{\sqrt{2} \cosh t}{(\sqrt{2})^3 \cosh^3 t} = \frac{1}{2} \operatorname{sech}^2 t;$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}} \tanh t & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \operatorname{sech} t \\ \operatorname{sech} t & 0 & -\tanh t \end{vmatrix} = \left(\frac{1}{\sqrt{2}} \tanh t\right)\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}} \operatorname{sech} t\right)\mathbf{k};$$

$$\frac{d\mathbf{a}}{dt} = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j} \Rightarrow \tau = \frac{\begin{vmatrix} \sinh t & -\cosh t & 1 \\ \cosh t & -\sinh t & 0 \\ \sinh t & -\cosh t & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{-1}{2 \cosh^2 t} = -\frac{1}{2} \operatorname{sech}^2 t$$

$$9. \mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + b\mathbf{k} \Rightarrow \mathbf{v} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + b\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-a \sin t)^2 + (a \cos t)^2 + b^2}$$

$$= \sqrt{a^2 + b^2} \Rightarrow a_T = \frac{d}{dt} |\mathbf{v}| = 0; \mathbf{a} = (-a \cos t)\mathbf{i} + (-a \sin t)\mathbf{j} \Rightarrow |\mathbf{a}| = \sqrt{(-a \cos t)^2 + (-a \sin t)^2} = \sqrt{a^2} = |a|$$

$$\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{a^2 - 0^2} = |a| = |a| \Rightarrow \mathbf{a} = (0)\mathbf{T} + |a|\mathbf{N} = |a|\mathbf{N}$$

$$10. \mathbf{r} = (1 + 3t)\mathbf{i} + (t - 2)\mathbf{j} - 3t\mathbf{k} \Rightarrow \mathbf{v} = 3\mathbf{i} + \mathbf{j} - 3\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{3^2 + 1^2 + (-3)^2} = \sqrt{19} \Rightarrow \mathbf{a}_T = \frac{d}{dt}|\mathbf{v}| = 0; \mathbf{a} = \mathbf{0} \\ \Rightarrow \mathbf{a}_N = \sqrt{|\mathbf{a}|^2 - \mathbf{a}_T^2} = 0 \Rightarrow \mathbf{a} = (0)\mathbf{T} + (0)\mathbf{N} = \mathbf{0}$$

$$11. \mathbf{r} = (t + 1)\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + 2^2 + (2t)^2} = \sqrt{5 + 4t^2} \Rightarrow \mathbf{a}_T = \frac{1}{2}(5 + 4t^2)^{-1/2}(8t) \\ = 4t(5 + 4t^2)^{-1/2} \Rightarrow \mathbf{a}_T(1) = \frac{4}{\sqrt{9}} = \frac{4}{3}; \mathbf{a} = 2\mathbf{k} \Rightarrow \mathbf{a}(1) = 2\mathbf{k} \Rightarrow |\mathbf{a}(1)| = 2 \Rightarrow \mathbf{a}_N = \sqrt{|\mathbf{a}|^2 - \mathbf{a}_T^2} = \sqrt{2^2 - \left(\frac{4}{3}\right)^2} \\ = \sqrt{\frac{20}{9}} = \frac{2\sqrt{5}}{3} \Rightarrow \mathbf{a}(1) = \frac{4}{3}\mathbf{T} + \frac{2\sqrt{5}}{3}\mathbf{N}$$

$$12. \mathbf{r} = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = (\cos t - t \sin t)\mathbf{i} + (\sin t + t \cos t)\mathbf{j} + 2t\mathbf{k} \\ \Rightarrow |\mathbf{v}| = \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + (2t)^2} = \sqrt{5t^2 + 1} \Rightarrow \mathbf{a}_T = \frac{1}{2}(5t^2 + 1)^{-1/2}(10t) \\ = \frac{5t}{\sqrt{5t^2 + 1}} \Rightarrow \mathbf{a}_T(0) = 0; \mathbf{a} = (-2 \sin t - t \cos t)\mathbf{i} + (2 \cos t - t \sin t)\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{j} + 2\mathbf{k} \Rightarrow |\mathbf{a}(0)| \\ = \sqrt{2^2 + 2^2} = 2\sqrt{2} \Rightarrow \mathbf{a}_N = \sqrt{|\mathbf{a}|^2 - \mathbf{a}_T^2} = \sqrt{(2\sqrt{2})^2 - 0^2} = 2\sqrt{2} \Rightarrow \mathbf{a}(0) = (0)\mathbf{T} + 2\sqrt{2}\mathbf{N} = 2\sqrt{2}\mathbf{N}$$

$$13. \mathbf{r} = t^2\mathbf{i} + \left(t + \frac{1}{3}t^3\right)\mathbf{j} + \left(t - \frac{1}{3}t^3\right)\mathbf{k} \Rightarrow \mathbf{v} = 2t\mathbf{i} + (1 + t^2)\mathbf{j} + (1 - t^2)\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(2t)^2 + (1 + t^2)^2 + (1 - t^2)^2} \\ = \sqrt{2(t^4 + 2t^2 + 1)} = \sqrt{2}(1 + t^2) \Rightarrow \mathbf{a}_T = 2t\sqrt{2} \Rightarrow \mathbf{a}_T(0) = 0; \mathbf{a} = 2\mathbf{i} + 2t\mathbf{j} - 2t\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{i} \Rightarrow |\mathbf{a}(0)| = 2 \\ \Rightarrow \mathbf{a}_N = \sqrt{|\mathbf{a}|^2 - \mathbf{a}_T^2} = \sqrt{2^2 - 0^2} = 2 \Rightarrow \mathbf{a}(0) = (0)\mathbf{T} + 2\mathbf{N} = 2\mathbf{N}$$

$$14. \mathbf{r} = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + \sqrt{2}e^t\mathbf{k} \Rightarrow \mathbf{v} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} + \sqrt{2}e^t\mathbf{k} \\ \Rightarrow |\mathbf{v}| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + (\sqrt{2}e^t)^2} = \sqrt{4e^{2t}} = 2e^t \Rightarrow \mathbf{a}_T = 2e^t \Rightarrow \mathbf{a}_T(0) = 2; \\ \mathbf{a} = (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t)\mathbf{i} + (e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t)\mathbf{j} + \sqrt{2}e^t\mathbf{k} \\ = (-2e^t \sin t)\mathbf{i} + (2e^t \cos t)\mathbf{j} + \sqrt{2}e^t\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{j} + \sqrt{2}\mathbf{k} \Rightarrow |\mathbf{a}(0)| = \sqrt{2^2 + (\sqrt{2})^2} = \sqrt{6} \\ \Rightarrow \mathbf{a}_N = \sqrt{|\mathbf{a}|^2 - \mathbf{a}_T^2} = \sqrt{(\sqrt{6})^2 - 2^2} = \sqrt{2} \Rightarrow \mathbf{a}(0) = 2\mathbf{T} + \sqrt{2}\mathbf{N}$$

$$15. \mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - \mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} \\ = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}; \frac{d\mathbf{T}}{dt} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\cos t)^2 + (-\sin t)^2} \\ = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} \Rightarrow \mathbf{N}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}; \mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \mathbf{k} \\ \Rightarrow \mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{k}, \text{ the normal to the osculating plane; } \mathbf{r}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} - \mathbf{k} \Rightarrow \mathbf{P} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1\right) \text{ lies on the}$$

osculating plane $\Rightarrow 0\left(x - \frac{\sqrt{2}}{2}\right) + 0\left(y - \frac{\sqrt{2}}{2}\right) + (z - (-1)) = 0 \Rightarrow z = -1$ is the osculating plane; $\mathbf{T}$ is normal to the normal plane $\Rightarrow \left(-\frac{\sqrt{2}}{2}\right)\left(x - \frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(y - \frac{\sqrt{2}}{2}\right) + 0(z - (-1)) = 0 \Rightarrow -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y = 0$
 $\Rightarrow -x + y = 0$ is the normal plane; $\mathbf{N}$ is normal to the rectifying plane
 $\Rightarrow \left(-\frac{\sqrt{2}}{2}\right)\left(x - \frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{2}}{2}\right)\left(y - \frac{\sqrt{2}}{2}\right) + 0(z - (-1)) = 0 \Rightarrow -\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y = -1 \Rightarrow x + y = \sqrt{2}$ is the rectifying plane

$$\begin{aligned}
 16. \quad \mathbf{r} &= (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} \\
 &= \left(-\frac{1}{\sqrt{2}} \sin t\right)\mathbf{i} + \left(\frac{1}{\sqrt{2}} \cos t\right)\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{1}{\sqrt{2}} \cos t\right)\mathbf{i} + \left(-\frac{1}{\sqrt{2}} \sin t\right)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| \\
 &= \sqrt{\frac{1}{2} \cos^2 t + \frac{1}{2} \sin^2 t} = \frac{1}{\sqrt{2}} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j}; \text{ thus } \mathbf{T}(0) = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \text{ and } \mathbf{N}(0) = -\mathbf{i} \\
 \Rightarrow \mathbf{B}(0) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \end{vmatrix} = -\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}, \text{ the normal to the osculating plane; } \mathbf{r}(0) = \mathbf{i} \Rightarrow P(1, 0, 0) \text{ lies on}
 \end{aligned}$$

the osculating plane $\Rightarrow 0(x - 1) - \frac{1}{\sqrt{2}}(y - 0) + \frac{1}{\sqrt{2}}(z - 0) = 0 \Rightarrow y - z = 0$ is the osculating plane; $\mathbf{T}$ is normal to the normal plane $\Rightarrow 0(x - 1) + \frac{1}{\sqrt{2}}(y - 0) + \frac{1}{\sqrt{2}}(z - 0) = 0 \Rightarrow y + z = 0$ is the normal plane; $\mathbf{N}$ is normal to the rectifying plane $\Rightarrow -1(x - 1) + 0(y - 0) + 0(z - 0) = 0 \Rightarrow x = 1$ is the rectifying plane

17. Yes. If the car is moving along a curved path, then $\kappa \neq 0$ and $a_N = \kappa |\mathbf{v}|^2 \neq 0 \Rightarrow \mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N} \neq \mathbf{0}$.

18. $|\mathbf{v}|$ constant $\Rightarrow a_T = \frac{d}{dt} |\mathbf{v}| = 0 \Rightarrow \mathbf{a} = a_N \mathbf{N}$ is orthogonal to $\mathbf{T} \Rightarrow$ the acceleration is normal to the path

19. $\mathbf{a} \perp \mathbf{v} \Rightarrow \mathbf{a} \perp \mathbf{T} \Rightarrow a_T = 0 \Rightarrow \frac{d}{dt} |\mathbf{v}| = 0 \Rightarrow |\mathbf{v}|$ is constant

$$\begin{aligned}
 20. \quad \mathbf{r}(t) &= x(t)\mathbf{i} + (x(t))^2\mathbf{j} \Rightarrow \mathbf{v}(t) = \frac{dx}{dt}\mathbf{i} + 2x(t)\frac{dx}{dt}\mathbf{j} \Rightarrow |\mathbf{v}(t)| = \left|\frac{dx}{dt}\right| \sqrt{1 + 4x^2} = 10 \\
 \Rightarrow \left|\frac{dx}{dt}\right| &= 10(1 + 4x^2)^{-1/2} \Rightarrow \frac{dx}{dt} = \pm 10(1 + 4x^2)^{-1/2} \\
 \mathbf{a}(t) &= \frac{d}{dt} \left(\frac{dx}{dt} \mathbf{i} + 2x \frac{dx}{dt} \mathbf{j} \right) = \frac{d^2x}{dt^2} \mathbf{i} + \left[2 \left(\frac{dx}{dt} \right)^2 + 2x \frac{d^2x}{dt^2} \right] \mathbf{j}; \frac{d^2x}{dt^2} = \mp 40x(1 + 4x^2)^{-3/2} \\
 \Rightarrow \mathbf{a}(t) &= \mp \frac{40x}{(1 + 4x^2)^{3/2}} \mathbf{i} + \left[\frac{200}{1 + 4x^2} \mp \frac{80x^2}{(1 + 4x^2)^{3/2}} \right] \mathbf{j}; \text{ At } x = 0, \mathbf{a}(t) = 200\mathbf{j} \Rightarrow \mathbf{F} = m\mathbf{a} = 200m\mathbf{j}; \\
 \text{At } x = \sqrt{2}, \mathbf{a}(t) &= \mp \frac{40\sqrt{2}}{27} \mathbf{i} + \left(\frac{200}{9} \mp \frac{160}{27} \right) \mathbf{j} = \mp \frac{40\sqrt{2}}{27} \mathbf{i} + \frac{600 \mp 160}{27} \mathbf{j} \Rightarrow \mathbf{F} = m\mathbf{a} = \frac{40m}{27} \left[\mp \sqrt{2} \mathbf{i} + (15 \mp 3)\mathbf{j} \right]
 \end{aligned}$$

$$21. \mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}, \text{ where } a_T = \frac{d}{dt} |\mathbf{v}| = \frac{d}{dt} (\text{constant}) = 0 \text{ and } a_N = \kappa |\mathbf{v}|^2 \Rightarrow \mathbf{F} = m\mathbf{a} = m\kappa |\mathbf{v}|^2 \mathbf{N} \Rightarrow |\mathbf{F}| = m\kappa |\mathbf{v}|^2 \\ = (m |\mathbf{v}|^2) \kappa, \text{ a constant multiple of the curvature } \kappa \text{ of the trajectory}$$

$$22. a_N = 0 \Rightarrow \kappa |\mathbf{v}|^2 = 0 \Rightarrow \kappa = 0 \text{ (since the particle is moving, we cannot have zero speed)} \Rightarrow \text{the curvature is zero} \\ \text{so the particle is moving along a straight line}$$

$$23. (a) \mathbf{r} = xi + f(x)j \Rightarrow \mathbf{v} = i + f'(x)j \Rightarrow \mathbf{a} = f''(x)j \Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = f''(x)\mathbf{k}$$

$$\Rightarrow |\mathbf{v} \times \mathbf{a}| = \sqrt{(f''(x))^2} = |f''(x)| \text{ and } |\mathbf{v}| = \sqrt{1^2 + [f'(x)]^2} = \sqrt{1 + [f'(x)]^2} \Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

$$= \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

$$(b) y = \ln(\cos x) \Rightarrow \frac{dy}{dx} = \left(\frac{1}{\cos x}\right)(-\sin x) = -\tan x \Rightarrow \frac{d^2y}{dx^2} = -\sec^2 x \Rightarrow \kappa = \frac{|-\sec^2 x|}{[1 + (-\tan x)^2]^{3/2}} = \frac{\sec^2 x}{|\sec^3 x|} \\ = \frac{1}{\sec x} = \cos x, \text{ since } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$(c) x = x_0 \text{ gives a point of inflection} \Rightarrow f''(x_0) = 0 \text{ (since } f \text{ is twice differentiable)} \Rightarrow \kappa = 0$$

$$24. (a) \mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j} = xi + yj \Rightarrow \mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} \Rightarrow \mathbf{a} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} \Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dot{x} & \dot{y} & 0 \\ \ddot{x} & \ddot{y} & 0 \end{vmatrix} = (\dot{x}\ddot{y} - \dot{y}\ddot{x})\mathbf{k}$$

$$\Rightarrow |\mathbf{v} \times \mathbf{a}| = |\dot{x}\ddot{y} - \dot{y}\ddot{x}| \text{ and } |\mathbf{v}| = \sqrt{\dot{x}^2 + \dot{y}^2} \Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$

$$(b) \mathbf{r}(t) = t\mathbf{i} + \ln(\sin t)\mathbf{j}, 0 < t < \pi \Rightarrow x = t \text{ and } y = \ln(\sin t) \Rightarrow \dot{x} = 1, \ddot{x} = 0; \dot{y} = \frac{\cos t}{\sin t} = \cot t, \ddot{y} = -\csc^2 t$$

$$\Rightarrow \kappa = \frac{|-\csc^2 t - 0|}{(1 + \cot^2 t)^{3/2}} = \frac{\csc^2 t}{\csc^3 t} = \sin t$$

$$(c) \mathbf{r}(t) = \tan^{-1}(\sinh t)\mathbf{i} + \ln(\cosh t)\mathbf{j} \Rightarrow x = \tan^{-1}(\sinh t) \text{ and } y = \ln(\cosh t) \Rightarrow \dot{x} = \frac{\cosh t}{1 + \sinh^2 t} = \frac{1}{\cosh t} \\ = \operatorname{sech} t, \ddot{x} = -\operatorname{sech} t \tanh t; \dot{y} = \frac{\sinh t}{\cosh t} = \tanh t, \ddot{y} = \operatorname{sech}^2 t \Rightarrow \kappa = \frac{|\operatorname{sech}^3 t + \operatorname{sech} t \tanh^2 t|}{(\operatorname{sech}^2 t + \tanh^2 t)} = |\operatorname{sech} t| \\ = \operatorname{sech} t$$

$$25. (a) \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \Rightarrow \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j} \text{ is tangent to the curve at the point } (f(t), g(t)); \\ \mathbf{n} \cdot \mathbf{v} = [-g'(t)\mathbf{i} + f'(t)\mathbf{j}] \cdot [f'(t)\mathbf{i} + g'(t)\mathbf{j}] = -g'(t)f'(t) + f'(t)g'(t) = 0; -\mathbf{n} \cdot \mathbf{v} = -(\mathbf{n} \cdot \mathbf{v}) = 0; \text{ thus,} \\ \mathbf{n} \text{ and } -\mathbf{n} \text{ are both normal to the curve at the point}$$

(b) $\mathbf{r}(t) = t\mathbf{i} + e^{2t}\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2e^{2t}\mathbf{j} \Rightarrow \mathbf{n} = -2e^{2t}\mathbf{i} + \mathbf{j}$ points toward the concave side of the curve; $\mathbf{N} = \frac{\mathbf{n}}{|\mathbf{n}|}$ and

$$|\mathbf{n}| = \sqrt{4e^{4t} + 1} \Rightarrow \mathbf{N} = \frac{-2e^{2t}}{\sqrt{1 + 4e^{4t}}}\mathbf{i} + \frac{1}{\sqrt{1 + 4e^{4t}}}\mathbf{j}$$

(c) $\mathbf{r}(t) = \sqrt{4-t^2}\mathbf{i} + t\mathbf{j} \Rightarrow \mathbf{v} = \frac{-t}{\sqrt{4-t^2}}\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{n} = -\mathbf{i} - \frac{t}{\sqrt{4-t^2}}\mathbf{j}$ points toward the concave side of the curve;

$$\mathbf{N} = \frac{\mathbf{n}}{|\mathbf{n}|} \text{ and } |\mathbf{n}| = \sqrt{1 + \frac{t^2}{4-t^2}} = \frac{2}{\sqrt{4-t^2}} \Rightarrow \mathbf{N} = -\frac{1}{2}(\sqrt{4-t^2}\mathbf{i} + t\mathbf{j})$$

26. (a) $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}t^3\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + t^2\mathbf{j} \Rightarrow \mathbf{n} = t^2\mathbf{i} - \mathbf{j}$ points toward the concave side of the curve when $t < 0$ and $-\mathbf{n} = -t^2\mathbf{i} + \mathbf{j}$ points toward the concave side when $t > 0 \Rightarrow \mathbf{N} = \frac{1}{\sqrt{1+t^4}}(t^2\mathbf{i} - \mathbf{j})$ for $t < 0$ and

$$\mathbf{N} = \frac{1}{\sqrt{1+t^4}}(-t^2\mathbf{i} + \mathbf{j}) \text{ for } t > 0$$

(b) From part (a), $|\mathbf{v}| = \sqrt{1+t^4} \Rightarrow \mathbf{T} = \frac{1}{\sqrt{1+t^4}}(\mathbf{i} + t^2\mathbf{j}) \Rightarrow \frac{d\mathbf{T}}{dt} = \frac{-2t^3}{(1+t^4)^{3/2}}(\mathbf{i} + t^2\mathbf{j}) + \frac{1}{\sqrt{1+t^4}}(2t\mathbf{j})$

$$= \frac{-2t^3}{(1+t^4)^{3/2}}\left[\mathbf{i} + t^2\mathbf{j} - \left(\frac{1+t^4}{t^2}\right)\mathbf{j}\right] = \frac{-2t^3}{(1+t^4)^{3/2}}\left(\mathbf{i} - \frac{1}{t^2}\mathbf{j}\right) \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \frac{2|t|^3}{(1+t^4)^{3/2}}\sqrt{1+\frac{1}{t^4}} = \frac{2|t|}{(1+t^4)^{3/2}}\sqrt{1+t^4}$$

$$= \frac{2|t|}{1+t^4}; \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(\frac{1+t^4}{2|t|}\right)\left[\frac{-2t^3}{(1+t^4)^{3/2}}\right]\left(-\mathbf{i} + \frac{1}{t^2}\mathbf{j}\right) = \frac{t}{|t|}\left(-\frac{t^2}{\sqrt{1+t^4}}\mathbf{i} + \frac{1}{\sqrt{1+t^4}}\mathbf{j}\right), t \neq 0. \text{ The normal}$$

$\mathbf{N}$ does not exist at $t = 0$, where the curve has a point of inflection; $\left.\frac{d\mathbf{T}}{dt}\right|_{t=0} = 0$ so the curvature $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$

$$= \left|\frac{d\mathbf{T}}{dt} \cdot \frac{dt}{ds}\right| = 0 \text{ at } t = 0 \Rightarrow \mathbf{N} = \frac{1}{\kappa} \frac{d\mathbf{T}}{ds} \text{ is undefined. Since } x = t \text{ and } y = \frac{1}{3}t^3 \Rightarrow y = \frac{1}{3}x^3, \text{ the curve is the}$$

cubic power curve which is concave down for $x = t < 0$ and concave up for $x = t > 0$.

27. $y = ax^2 \Rightarrow y' = 2ax \Rightarrow y'' = 2a$; from Exercise 23(a), $\kappa(x) = \frac{|2a|}{(1+4a^2x^2)^{3/2}} = |2a|(1+4a^2x^2)^{-3/2}$

$\Rightarrow \kappa'(x) = -\frac{3}{2}|2a|(1+4a^2x^2)^{-5/2}(8a^2x)$; thus, $\kappa'(x) = 0 \Rightarrow x = 0$. Now, $\kappa'(x) > 0$ for $x < 0$ and $\kappa'(x) < 0$ for $x > 0$ so that $\kappa(x)$ has an absolute maximum at $x = 0$ which is the vertex of the parabola. Since $x = 0$ is the only critical point for $\kappa(x)$, the curvature has no minimum value.

28. $\mathbf{r} = (a \cos t)\mathbf{i} + (b \sin t)\mathbf{j} \Rightarrow \mathbf{v} = (-a \sin t)\mathbf{i} + (b \cos t)\mathbf{j} \Rightarrow \mathbf{a} = (-a \cos t)\mathbf{i} - (b \sin t)\mathbf{j} \Rightarrow \mathbf{v} \times \mathbf{a}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin t & b \cos t & 0 \\ -a \cos t & -b \sin t & 0 \end{vmatrix} = ab\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = |ab| = ab, \text{ since } a > b > 0; \kappa(t) = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

$$= ab(a^2 \sin^2 t + b^2 \cos^2 t)^{-3/2}; \kappa'(t) = -\frac{3}{2}(ab)(a^2 \sin^2 t + b^2 \cos^2 t)^{-5/2}(2a^2 \sin t \cos t - 2b^2 \sin t \cos t)$$

$$= -\frac{3}{2}(ab)(a^2 - b^2)(\sin 2t)(a^2 \sin^2 t + b^2 \cos^2 t)^{-5/2}; \text{ thus, } \kappa'(t) = 0 \Rightarrow \sin 2t = 0 \Rightarrow t = 0, \pi \text{ identifying}$$

points on the major axis, or $t = \frac{\pi}{2}, \frac{3\pi}{2}$ identifying points on the minor axis. Furthermore, $\kappa'(t) < 0$ for

$0 < t < \frac{\pi}{2}$ and for $\pi < t < \frac{3\pi}{2}$; $\kappa'(t) > 0$ for $\frac{\pi}{2} < t < \pi$ and $\frac{3\pi}{2} < t < 2\pi$. Therefore, the points associated with $t = 0$ and $t = \pi$ on the major axis give absolute maximum curvature and the points associated with $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ on the minor axis give absolute minimum curvature.

$$29. \kappa = \frac{a}{a^2 + b^2} \Rightarrow \frac{d\kappa}{da} = \frac{-a^2 + b^2}{(a^2 + b^2)^2}; \frac{d\kappa}{da} = 0 \Rightarrow a^2 + b^2 = 0 \Rightarrow a = \pm b \Rightarrow a = b \text{ since } a, b > 0. \text{ Now, } \frac{d\kappa}{da} > 0 \text{ if } a < b \text{ and } \frac{d\kappa}{da} < 0 \text{ if } a > b \Rightarrow \kappa \text{ is at a maximum for } a = b \text{ and } \kappa(b) = \frac{b}{b^2 + b^2} = \frac{1}{2b} \text{ is the maximum value.}$$

$$30. \text{ From Example 3, } |\mathbf{v}| = t \text{ and } a_N = t \text{ so that } a_N = \kappa |\mathbf{v}|^2 \Rightarrow \kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{t}{t^2} = \frac{1}{t}, t \neq 0 \Rightarrow \rho = \frac{1}{\kappa} = t$$

$$31. \mathbf{r} = (x_0 + At)\mathbf{i} + (y_0 + Bt)\mathbf{j} + (z_0 + Ct)\mathbf{k} \Rightarrow \mathbf{v} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k} \Rightarrow \mathbf{a} = \mathbf{0} \Rightarrow \mathbf{v} \times \mathbf{a} = \mathbf{0} \Rightarrow \kappa = 0. \text{ Since the curve is a plane curve, } \tau = 0.$$

$$32. \text{ From Example 4, the curvature of the helix } \mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + b t \mathbf{k}, a, b \geq 0 \text{ is } \kappa = \frac{a}{a^2 + b^2}; \text{ also } |\mathbf{v}| = \sqrt{a^2 + b^2}. \text{ For the helix } \mathbf{r}(t) = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 4\pi, a = 3 \text{ and } b = 1 \Rightarrow \kappa = \frac{3}{3^2 + 1^2} = \frac{3}{10} \text{ and } |\mathbf{v}| = \sqrt{10} \Rightarrow K = \int_0^{4\pi} \frac{3}{10} \sqrt{10} dt = \left[\frac{3}{\sqrt{10}} t \right]_0^{4\pi} = \frac{12\pi}{\sqrt{10}}$$

$$33. (a) \text{ From Exercise 30, } \kappa = \frac{1}{t} \text{ and } |\mathbf{v}| = t \Rightarrow K = \int_a^b \left(\frac{1}{t} \right) (t) dt = b - a$$

$$(b) y = x^2 \Rightarrow x = t \text{ and } y = t^2, -\infty < t < \infty \Rightarrow \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1 + 4t^2}; \text{ also } \mathbf{a} = 2\mathbf{j}$$

$$\Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = 2\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = 2 \Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{2}{(\sqrt{1 + 4t^2})^3}. \text{ Then}$$

$$K = \int_{-\infty}^{\infty} \frac{2}{(\sqrt{1 + 4t^2})^3} (\sqrt{1 + 4t^2}) dt = \int_{-\infty}^{\infty} \frac{2}{1 + 4t^2} dt = \lim_{a \rightarrow -\infty} \int_a^0 \frac{2}{1 + 4t^2} dt + \lim_{b \rightarrow \infty} \int_0^b \frac{2}{1 + 4t^2} dt$$

$$= \lim_{a \rightarrow -\infty} [\tan^{-1} 2t]_a^0 + \lim_{b \rightarrow \infty} [\tan^{-1} 2t]_0^b = \lim_{a \rightarrow -\infty} (-\tan^{-1} 2a) + \lim_{b \rightarrow \infty} (\tan^{-1} 2b) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$34. \text{ From Example 4, } \tau = \frac{b}{a^2 + b^2} \Rightarrow \tau'(b) = \frac{a^2 - b^2}{(a^2 + b^2)^2}; \tau'(b) = 0 \Rightarrow \frac{a^2 - b^2}{(a^2 + b^2)^2} = 0 \Rightarrow a^2 - b^2 = 0 \Rightarrow b = \pm a$$

$$\Rightarrow b = a \text{ since } a, b > 0. \text{ Also } b < a \Rightarrow \tau' > 0 \text{ and } b > a \Rightarrow \tau' < 0 \text{ so } \tau_{\max} \text{ occurs when } b = a \Rightarrow \tau_{\max} = \frac{a}{a^2 + a^2} = \frac{1}{2a}$$

$$35. \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \Rightarrow \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}; \mathbf{v} \cdot \mathbf{k} = 0 \Rightarrow h'(t) = 0 \Rightarrow h(t) = C$$

$$\Rightarrow \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + C\mathbf{k} \text{ and } \mathbf{r}(a) = f(a)\mathbf{i} + g(a)\mathbf{j} + C\mathbf{k} = \mathbf{0} \Rightarrow f(a) = 0, g(a) = 0 \text{ and } C = 0 \Rightarrow h(t) = 0.$$

36. From Example 4, $\mathbf{v} = -(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + b\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{a^2 + b^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

$$= \frac{1}{\sqrt{a^2 + b^2}} [-(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + b\mathbf{k}]; \frac{d\mathbf{T}}{dt} = \frac{1}{\sqrt{a^2 + b^2}} [-(a \cos t)\mathbf{i} - (a \sin t)\mathbf{j}] \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|}$$

$$= -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}; \mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{a \sin t}{\sqrt{a^2 + b^2}} & \frac{a \cos t}{\sqrt{a^2 + b^2}} & \frac{b}{\sqrt{a^2 + b^2}} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

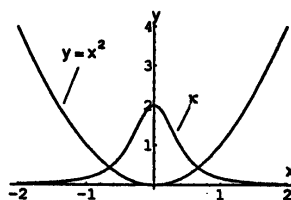
$$= \frac{b \sin t}{\sqrt{a^2 + b^2}}\mathbf{i} - \frac{b \cos t}{\sqrt{a^2 + b^2}}\mathbf{j} + \frac{a}{\sqrt{a^2 + b^2}}\mathbf{k} \Rightarrow \frac{d\mathbf{B}}{dt} = \frac{1}{\sqrt{a^2 + b^2}} [(b \cos t)\mathbf{i} + (b \sin t)\mathbf{j}] \Rightarrow \frac{d\mathbf{B}}{dt} \cdot \mathbf{N} = -\frac{b}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow \tau = -\frac{1}{|\mathbf{v}|} \left(\frac{d\mathbf{B}}{dt} \cdot \mathbf{N} \right) = \left(-\frac{1}{\sqrt{a^2 + b^2}} \right) \left(-\frac{b}{\sqrt{a^2 + b^2}} \right) = \frac{b}{a^2 + b^2}, \text{ which is consistent with the result in}$$

Example 4.

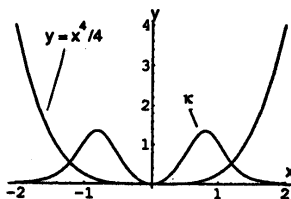
37. $y = x^2 \Rightarrow f'(x) = 2x$ and $f''(x) = 2$

$$\Rightarrow \kappa = \frac{|2|}{(1 + (2x)^2)^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$$



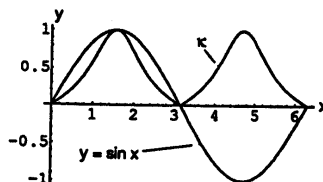
38. $y = \frac{x^4}{4} \Rightarrow f'(x) = x^3$ and $f''(x) = 3x^2$

$$\Rightarrow \kappa = \frac{|3x^2|}{(1 + (x^3)^2)^{3/2}} = \frac{3x^2}{(1 + x^6)^{3/2}}$$



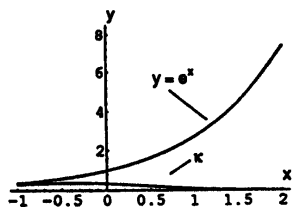
39. $y = \sin x \Rightarrow f'(x) = \cos x$ and $f''(x) = -\sin x$

$$\Rightarrow \kappa = \frac{|-\sin x|}{(1 + \cos^2 x)^{3/2}} = \frac{|\sin x|}{(1 + \cos^2 x)^{3/2}}$$



40. $y = e^x \Rightarrow f'(x) = e^x$ and $f''(x) = e^x$

$$\Rightarrow \kappa = \frac{|e^x|}{(1 + (e^x)^2)^{3/2}} = \frac{e^x}{(1 + e^{2x})^{3/2}}$$



41-48. Example CAS commands:

Maple:

```
with(plots):
x:= t -> t^3 - 2*t^2 - t;
y:= t -> 3*t/sqrt(1+t^2);
dx:= t -> D(x)(t);
dy:= t -> D(y)(t);
ds:= t -> sqrt((dx^2)(t) + (dy^2)(t));
d2x:= t -> D(dx)(t);
d2y:= t -> D(dy)(t);
kap:= t -> abs(dx(t)*d2y(t) - dy(t)*d2x(t))/((ds)(t))^3;
a:= t -> x0 - (1/kap(t))*(dy(t)/ds(t));
b:= t -> x0 + (1/kap(t))*(dx(t)/ds(t));
s1:= plot([x(t),y(t), t = -2..5], -15..5, -10..4, scaling=CONSTRAINED);
display(s1);
t0:=1: x0:= x(t0): y0:= y(t0):
circle:= ((x-a(t0))^2 + (y-b(t0))^2 = (1/kap(t0))^2);
s2:=implicitplot(circle, x=-15..6,y=-10..5,scaling=CONSTRAINED);
s3:=plot([a(t0),b(t0),x0,y0]);
display({s1,s2,s3});
```

Mathematica:

```
Clear[x,y,t]
r[t_] = {x[t],y[t]}
x[t_] = t^3 - 2 t^2 - t
y[t_] = 3 t / Sqrt[1+t^2]
{a,b} = {-2,5};
t0 = 1;
p1 = ParametricPlot[ {x[t],y[t]}, {t,a,b},
AspectRatio -> Automatic ]
v0 = r'[t0]
s0 = Sqrt[ v0 . v0 ]
k0 = Abs[ x' [t0] y''[t0] - y' [t0] x'' [t0] ]/s0^3
N[%]
n0 = { -y' [t0], x' [t0] } / s0
r0 = r[t0]
c0 = r0 + 1/k0 n0
```

Note: Plot the circle parametrically rather than implicitly:

```

circ = ParametricPlot[ Evaluate[c0 + 1/k0 {Cos[t],Sin[t]}],
  {t,0,2Pi}, AspectRatio -> Automatic ]
line = Graphics[{Line[{c0,r0}]}]
Show[ p1, circ, line ]

```

10.8 PLANETARY MOTION AND SATELLITES

- $$\frac{T^2}{a^3} = \frac{4\pi^2}{GM} \Rightarrow T^2 = \frac{4\pi^2}{GM} a^3 \Rightarrow T^2 = \frac{4\pi^2}{(6.6720 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2})(5.975 \times 10^{24} \text{ kg})} (6,808,000 \text{ m})^3$$

$$\approx 3.125 \times 10^7 \text{ sec}^2 \Rightarrow T \approx \sqrt{3.125 \times 10^7 \text{ sec}^2} \approx 55.90 \times 10^2 \text{ sec} \approx 93.2 \text{ min}$$
- $$e = 0.0167 \text{ and perihelion distance} = 149,577,000 \text{ km and } e = \frac{r_0 v_0^2}{GM} - 1$$

$$\Rightarrow 0.0167 = \frac{(149,577,000,000 \text{ m})v_0^2}{(6.6720 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2})(1.99 \times 10^{30} \text{ kg})} - 1 \Rightarrow v_0^2 \approx 9.02 \times 10^8 \text{ m}^2/\text{sec}^2$$

$$\Rightarrow v_0 \approx \sqrt{9.02 \times 10^8 \text{ m}^2/\text{sec}^2} \approx 3.00 \times 10^4 \text{ m/sec}$$
- $$92.25 \text{ min} = 5535 \text{ sec and } \frac{T^2}{a^3} = \frac{4\pi^2}{GM} \Rightarrow a^3 = \frac{GM}{4\pi^2} T^2$$

$$\Rightarrow a^3 = \frac{(6.6720 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2})(5.975 \times 10^{24} \text{ kg})}{4\pi^2} (5535 \text{ sec})^2 = 3.094 \times 10^{20} \text{ m}^3 \Rightarrow a \approx \sqrt[3]{3.094 \times 10^{20} \text{ m}^3}$$

$$= 6.763 \times 10^6 \text{ m} \approx 6763 \text{ km; the mean distance from center of the Earth} = \frac{12,757 \text{ km} + 183 \text{ km} + 589 \text{ km}}{2}$$

$$= 6765 \text{ km}$$
- $$T = 1639 \text{ min} = 98,340 \text{ sec and mass of Mars} = 6.418 \times 10^{23} \text{ kg} \Rightarrow a^3 = \frac{GM}{4\pi^2} T^2$$

$$= \frac{(6.6720 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2})(6.418 \times 10^{23} \text{ kg})(98,340 \text{ sec})^2}{4\pi^2} \approx 1.049 \times 10^{22} \text{ m}^3 \Rightarrow a \approx \sqrt[3]{1.049 \times 10^{22} \text{ m}^3}$$

$$= 2.189 \times 10^7 \text{ m} = 21,890 \text{ km}$$
- $$2a = \text{diameter of Mars} + \text{perigee height} + \text{apogee height} = D + 1499 \text{ km} + 35,800 \text{ km}$$

$$\Rightarrow 2(21,890) \text{ km} = D + 37,300 \text{ km} \Rightarrow D = 6480 \text{ km}$$
- $$a = 22,030 \text{ km} = 2.203 \times 10^7 \text{ m and } T^2 = \frac{4\pi^2}{GM} a^3$$

$$\Rightarrow T^2 = \frac{4\pi^2}{(6.6720 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2})(6.418 \times 10^{23} \text{ kg})} (2.203 \times 10^7 \text{ sec})^3 \approx 9.857 \times 10^9 \text{ sec}^2$$

$$\Rightarrow T \approx \sqrt{9.857 \times 10^9 \text{ sec}^2} \approx 9.928 \times 10^4 \text{ sec} \approx 1655 \text{ min}$$
- $$(a) \text{ Period of the satellite} = \text{rotational period of the Earth} \Rightarrow \text{period of the satellite} = 1436.1 \text{ min}$$

$$= 86,166 \text{ sec; } a^3 = \frac{GMT^2}{4\pi^2} \Rightarrow a^3 = \frac{(6.6720 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2})(5.975 \times 10^{24} \text{ kg})(86,166 \text{ sec})^2}{4\pi^2}$$

$$\approx 7.4973 \times 10^{22} \text{ m}^3 \Rightarrow a \approx \sqrt[3]{74.973 \times 10^{21} \text{ m}^3} \approx 4.2167 \times 10^7 \text{ m} = 42,167 \text{ km}$$

(b) The radius of the Earth is approximately 6379 km $\Rightarrow$ the height of the orbit is $42,167 - 6379 = 35,788 \text{ km}$

(c) Symcom 3, GOES 4, and Intelsat 5

$$\begin{aligned} 8. \quad T &= 1477.4 \text{ min} = 88,644 \text{ sec} \Rightarrow a^3 = \frac{GMT^2}{4\pi^2} \\ &= \frac{(6.6720 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2})(6.418 \times 10^{23} \text{ kg})(88,644 \text{ sec})^2}{4\pi^2} = 8.523 \times 10^{21} \text{ m}^3 \Rightarrow a \approx \sqrt[3]{8.523 \times 10^{21} \text{ m}^3} \\ &\approx 2.043 \times 10^7 \text{ m} = 20,430 \text{ km} \end{aligned}$$

$$\begin{aligned} 9. \quad \text{Period of the Moon} &= 2.36055 \times 10^6 \text{ sec} \Rightarrow a^3 = \frac{GMT^2}{4\pi^2} \\ &= \frac{(6.6720 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2})(5.975 \times 10^{24} \text{ kg})(2.36055 \times 10^6 \text{ sec})^2}{4\pi^2} \approx 5.627 \times 10^{25} \text{ m}^3 \Rightarrow a \approx \sqrt[3]{5.627 \times 10^{25} \text{ m}^3} \\ &\approx 3.832 \times 10^8 \text{ m} = 383,200 \text{ km from the center of the Earth, or about } 376,821 \text{ km from the surface} \end{aligned}$$

$$10. \quad r = \frac{GM}{v^2} \Rightarrow v^2 = \frac{GM}{r} \Rightarrow |v| = \sqrt{\frac{GM}{r}} = \sqrt{\frac{(6.6720 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2})(5.975 \times 10^{24} \text{ kg})}{r}} \approx 1.9966 \times 10^7 r^{-1/2} \text{ m/sec}$$

$$11. \quad \text{Solar System: } \frac{T^2}{a^3} = \frac{4\pi^2}{(6.6720 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2})(1.99 \times 10^{30} \text{ kg})} \approx 2.97 \times 10^{-19} \text{ sec}^2/\text{m}^3;$$

$$\text{Earth: } \frac{T^2}{a^3} = \frac{4\pi^2}{(6.6720 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2})(5.975 \times 10^{24} \text{ kg})} \approx 9.903 \times 10^{-14} \text{ sec}^2/\text{m}^3;$$

$$\text{Moon: } \frac{T^2}{a^3} = \frac{4\pi^2}{(6.6720 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2})(7.354 \times 10^{22} \text{ kg})} \approx 8.046 \times 10^{-12} \text{ sec}^2/\text{m}^3;$$

$$12. \quad e = \frac{r_0 v_0^2}{GM} - 1 \Rightarrow v_0^2 = \frac{GM(e+1)}{r_0} \Rightarrow v_0 = \sqrt{\frac{GM(e+1)}{r_0}};$$

$$\text{Circle: } e = 0 \Rightarrow v_0 = \sqrt{\frac{GM}{r_0}}$$

$$\text{Ellipse: } 0 < e < 1 \Rightarrow \sqrt{\frac{GM}{r_0}} < v_0 < \sqrt{\frac{2GM}{r_0}}$$

$$\text{Parabola: } e = 1 \Rightarrow v_0 = \sqrt{\frac{2GM}{r_0}}$$

$$\text{Hyperbola: } e > 1 \Rightarrow v_0 > \sqrt{\frac{2GM}{r_0}}$$

$$13. \quad r = \frac{GM}{v^2} \Rightarrow v^2 = \frac{GM}{r} \Rightarrow v = \sqrt{\frac{GM}{r}} \text{ which is constant since } G, M, \text{ and } r \text{ (the radius of orbit) are constant}$$

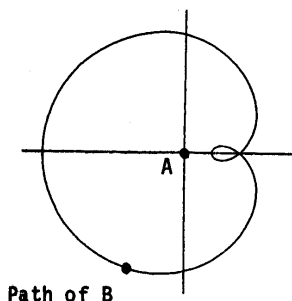
$$\begin{aligned} 14. \quad \Delta A &= \frac{1}{2} |\mathbf{r}(t + \Delta t) \times \mathbf{r}(t)| \Rightarrow \frac{\Delta A}{\Delta t} = \frac{1}{2} \left| \frac{\mathbf{r}(t + \Delta t) \times \mathbf{r}(t)}{\Delta t} \right| = \frac{1}{2} \left| \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t) + \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t) \right| \\ &= \frac{1}{2} \left| \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t) + \frac{1}{\Delta t} \mathbf{r}(t) \times \mathbf{r}(t) \right| = \frac{1}{2} \left| \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t) \right| \Rightarrow \frac{dA}{dt} = \lim_{\Delta t \rightarrow 0} \frac{1}{2} \left| \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \times \mathbf{r}(t) \right| \end{aligned}$$

$$= \frac{1}{2} \left| \frac{d\mathbf{r}}{dt} \times \mathbf{r}(t) \right| = \frac{1}{2} \left| \mathbf{r}(t) \times \frac{d\mathbf{r}}{dt} \right| = \frac{1}{2} |\mathbf{r} \times \dot{\mathbf{r}}|$$

$$\begin{aligned} 15. \quad T &= \left(\frac{2\pi a^2}{r_0^2 v_0^2} \right) \sqrt{1-e^2} \Rightarrow T^2 = \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2} \right) (1-e^2) = \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2} \right) \left[1 - \left(\frac{r_0 v_0^2}{GM} - 1 \right)^2 \right] \text{ (from Equation 34)} \\ &= \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2} \right) \left[-\frac{r_0^2 v_0^4}{G^2 M^2} + 2 \left(\frac{r_0 v_0^2}{GM} \right) \right] = \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2} \right) \left[\frac{2GM r_0 v_0^2 - r_0^2 v_0^4}{G^2 M^2} \right] = \frac{(4\pi^2 a^4)(2GM - r_0 v_0^2)}{r_0 G^2 M^2} \\ &= (4\pi^2 a^4) \left(\frac{2GM - r_0 v_0^2}{2r_0 GM} \right) \left(\frac{2}{GM} \right) = (4\pi^2 a^4) \left(\frac{1}{2a} \right) \left(\frac{2}{GM} \right) \text{ (from Equation 35)} \Rightarrow T^2 = \frac{4\pi^2 a^3}{GM} \Rightarrow \frac{T^2}{a^3} = \frac{4\pi^2}{GM} \end{aligned}$$

16. Let $\mathbf{r}_{AB}(t)$ denote the vector from planet A to planet B at time t . Then $\mathbf{r}_{AB}(t) = \mathbf{r}_B(t) - \mathbf{r}_A(t)$
- $$\begin{aligned} &= [3 \cos(\pi t) - 2 \cos(2\pi t)]\mathbf{i} + [3 \sin(\pi t) - 2 \sin(2\pi t)]\mathbf{j} \\ &= [3 \cos(\pi t) - 2(\cos^2(\pi t) - \sin^2(\pi t))]\mathbf{i} + [3 \sin(\pi t) - 4 \sin(\pi t) \cos(\pi t)]\mathbf{j} \\ &= [3 \cos(\pi t) - 4 \cos^2(\pi t) + 2]\mathbf{i} + [(3 - 4 \cos(\pi t)) \sin(\pi t)]\mathbf{j} \Rightarrow \text{parametric equations for the path are} \\ &\mathbf{x}(t) = 2 + [3 - 4 \cos(\pi t)] \cos(\pi t) \text{ and } \mathbf{y}(t) = [3 - 4 \cos(\pi t)] \sin(\pi t) \end{aligned}$$

17. Setting $\theta = \pi t$ and $r = 3 - 4 \cos \theta$, we see that
- $$x - 2 = r \cos \theta \text{ and } y = r \sin \theta \Rightarrow \text{the graph}$$
- of the path of planet B is the limaçon
- $$r = 3 - 4 \cos \theta \text{ shown at the right. The}$$
- planet A is located at $x = -2$.



18. (i) Perihelion is the time t such that $|\mathbf{r}(t)|$ is a minimum.
 (ii) Aphelion is the time t such that $|\mathbf{r}(t)|$ is a maximum.
 (iii) Equinox is the time t such that $\mathbf{r}(t) \cdot \mathbf{w} = 0$.
 (iv) Summer solstice is the time t such that the angle between $\mathbf{r}(t)$ and $\mathbf{w}$ is a maximum.
 (v) Winter solstice is the time t such that the angle between $\mathbf{r}(t)$ and $\mathbf{w}$ is a minimum.

CHAPTER 10 PRACTICE EXERCISES

1. $\text{length} = |2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}| = \sqrt{4 + 9 + 36} = 7$, $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k} = 7\left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right) \Rightarrow \text{the direction is } \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$

$$2. \text{ length} = |\mathbf{i} + 2\mathbf{j} - \mathbf{k}| = \sqrt{1+4+1} = \sqrt{6}, \mathbf{i} + 2\mathbf{j} - \mathbf{k} = \sqrt{6} \left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k} \right) \Rightarrow \text{the direction is}$$

$$\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

$$3. 2 \frac{\mathbf{v}}{|\mathbf{v}|} = 2 \cdot \frac{4\mathbf{i} - \mathbf{j} + 4\mathbf{k}}{\sqrt{4^2 + (-1)^2 + 4^2}} = 2 \cdot \frac{4\mathbf{i} - \mathbf{j} + 4\mathbf{k}}{\sqrt{33}} = \frac{8}{\sqrt{33}}\mathbf{i} - \frac{2}{\sqrt{33}}\mathbf{j} + \frac{8}{\sqrt{33}}\mathbf{k}$$

$$4. -5 \frac{\mathbf{v}}{|\mathbf{v}|} = -5 \cdot \frac{\left(\frac{3}{5}\right)\mathbf{i} + \left(\frac{4}{5}\right)\mathbf{j}}{\sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2}} = -5 \cdot \frac{\left(\frac{3}{5}\right)\mathbf{i} + \left(\frac{4}{5}\right)\mathbf{j}}{\sqrt{\frac{9}{25} + \frac{16}{25}}} = -3\mathbf{i} - 4\mathbf{j}$$

$$5. |\mathbf{v}| = \sqrt{1+1} = \sqrt{2}, |\mathbf{u}| = \sqrt{4+1+4} = 3, \mathbf{v} \cdot \mathbf{u} = 3, \mathbf{u} \cdot \mathbf{v} = 3, \mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & 1 & -2 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j} - \mathbf{k},$$

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}, |\mathbf{v} \times \mathbf{u}| = \sqrt{4+4+1} = 3, \theta = \cos^{-1} \left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}||\mathbf{u}|} \right) = \cos^{-1} \left(\frac{1}{\sqrt{2}} \right) = \frac{\pi}{4},$$

$$|\mathbf{u}| \cos \theta = \frac{3}{\sqrt{2}}, \text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} = \frac{3}{2}(\mathbf{i} + \mathbf{j})$$

$$6. |\mathbf{v}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}, |\mathbf{u}| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}, \mathbf{v} \cdot \mathbf{u} = (1)(-1) + (1)(0) + (2)(-1) = -3,$$

$$\mathbf{u} \cdot \mathbf{v} = -3, \mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{vmatrix} = -\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) = \mathbf{i} + \mathbf{j} - \mathbf{k},$$

$$|\mathbf{v} \times \mathbf{u}| = \sqrt{(-1)^2 + (-1)^2 + 1^2} = \sqrt{3}, \theta = \cos^{-1} \left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}||\mathbf{u}|} \right) = \cos^{-1} \left(\frac{-3}{\sqrt{6}\sqrt{2}} \right) = \cos^{-1} \left(\frac{-3}{\sqrt{12}} \right)$$

$$= \cos^{-1} \left(-\frac{\sqrt{3}}{2} \right) = \frac{5\pi}{6}, |\mathbf{u}| \cos \theta = \frac{-3}{\sqrt{6}} = -\sqrt{\frac{9}{6}} = -\sqrt{\frac{3}{2}}, \text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} = \frac{-3}{6}(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = -\frac{1}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

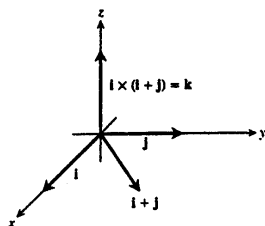
$$7. \mathbf{u} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} + \left[\mathbf{u} - \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} \right] = \frac{4}{3}(2\mathbf{i} + \mathbf{j} - \mathbf{k}) + \left[(\mathbf{i} + \mathbf{j} - 5\mathbf{k}) - \frac{4}{3}(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \right] = \frac{4}{3}(2\mathbf{i} + \mathbf{j} - \mathbf{k}) - \frac{1}{3}(5\mathbf{i} + \mathbf{j} + 11\mathbf{k}),$$

where $\mathbf{v} \cdot \mathbf{u} = 8$ and $\mathbf{v} \cdot \mathbf{v} = 6$

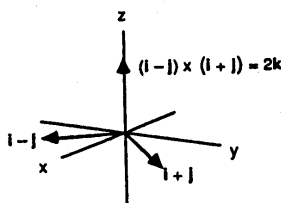
$$8. \mathbf{u} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} + \left[\mathbf{u} - \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} \right] = -\frac{1}{5}(\mathbf{i} - 2\mathbf{j}) + \left[(\mathbf{i} + \mathbf{j} + \mathbf{k}) - \left(\frac{-1}{5} \right)(\mathbf{i} - 2\mathbf{j}) \right] = -\frac{1}{5}(\mathbf{i} - 2\mathbf{j}) + \left(\frac{6}{5}\mathbf{i} + \frac{3}{5}\mathbf{j} + \mathbf{k} \right),$$

where $\mathbf{v} \cdot \mathbf{u} = -1$ and $\mathbf{v} \cdot \mathbf{v} = 5$

$$9. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \mathbf{k}$$



$$10. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 2\mathbf{k}$$



$$\begin{aligned} 11. \text{ Let } \mathbf{v} &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \text{ and } \mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}. \text{ Then } |\mathbf{v} - 2\mathbf{w}|^2 = |(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) - 2(w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k})|^2 \\ &= |(v_1 - 2w_1)\mathbf{i} + (v_2 - 2w_2)\mathbf{j} + (v_3 - 2w_3)\mathbf{k}|^2 = \left(\sqrt{(v_1 - 2w_1)^2 + (v_2 - 2w_2)^2 + (v_3 - 2w_3)^2} \right)^2 \\ &= (v_1^2 + v_2^2 + v_3^2) - 4(v_1w_1 + v_2w_2 + v_3w_3) + 4(w_1^2 + w_2^2 + w_3^2) = |\mathbf{v}|^2 - 4\mathbf{v} \cdot \mathbf{w} + 4|\mathbf{w}|^2 \\ &= |\mathbf{v}|^2 - 4|\mathbf{u}||\mathbf{w}|\cos\theta + 4|\mathbf{w}|^2 = 4 - 4(2)(3)\left(\cos\frac{\pi}{3}\right) + 36 = 40 - 24\left(\frac{1}{2}\right) = 40 - 12 = 28 \Rightarrow |\mathbf{v} - 2\mathbf{w}| = \sqrt{28} \\ &= 2\sqrt{7} \end{aligned}$$

$$12. \mathbf{u} \text{ and } \mathbf{v} \text{ are parallel when } \mathbf{u} \times \mathbf{v} = \mathbf{0} \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -5 \\ -4 & -8 & a \end{vmatrix} = \mathbf{0} \Rightarrow (4a - 40)\mathbf{i} + (20 - 2a)\mathbf{j} + (0)\mathbf{k} = \mathbf{0}$$

$$\Rightarrow 4a - 40 = 0 \text{ and } 20 - 2a = 0 \Rightarrow a = 10$$

$$13. (a) \text{ area} = |\mathbf{u} \times \mathbf{v}| = \text{abs} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix} = |2\mathbf{i} - 3\mathbf{j} - \mathbf{k}| = \sqrt{4 + 9 + 1} = \sqrt{14}$$

$$(b) \text{ volume} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ -1 & -2 & 3 \end{vmatrix} = 1(3 + 2) + 1(-1 - 6) - 1(-4 + 1) = 1$$

$$14. \text{ (a) area} = |\mathbf{u} \times \mathbf{v}| = \text{abs} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = |\mathbf{k}| = 1$$

$$\text{ (b) volume} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1(1-0) + 1(0-0) + 0 = 1$$

15. The desired vector is $\mathbf{n} \times \mathbf{v}$ or $\mathbf{v} \times \mathbf{n}$ since $\mathbf{n} \times \mathbf{v}$ is perpendicular to both $\mathbf{n}$ and $\mathbf{v}$ and, therefore, also parallel to the plane.

16. If $a = 0$ and $b \neq 0$, then the line $by = c$ and $\mathbf{i}$ are parallel. If $a \neq 0$ and $b = 0$, then the line $ax = c$ and $\mathbf{j}$ are parallel. If a and b are both $\neq 0$, then $ax + by = c$ contains the points $(\frac{c}{a}, 0)$ and $(0, \frac{c}{b}) \Rightarrow$ the vector $ab(\frac{c}{a}\mathbf{i} - \frac{c}{b}\mathbf{j}) = c(b\mathbf{i} - a\mathbf{j})$ and the line are parallel. Therefore, the vector $b\mathbf{i} - a\mathbf{j}$ is parallel to the line $ax + by = c$ in every case.

17. Parametric equations for the line are $x = 1 - 3t$, $y = 2$, $z = 3 + 7t$.

18. The line is parallel to $\vec{PQ} = 0\mathbf{i} + \mathbf{j} - \mathbf{k}$ and contains the point $P(1, 2, 0) \Rightarrow$ parametric equations are $x = 1$, $y = 2 + t$, $z = -t$ for $0 \leq t \leq 1$.

19. $P(3, -2, 1)$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow (2)(x-3) + (1)(y-(-2)) + (1)(z-1) = 0 \Rightarrow 2x + y + z = 5$

20. $P(-1, 6, 0)$ and $\mathbf{n} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \Rightarrow (1)(x-(-1)) + (-2)(y-6) + (3)(z-0) = 0 \Rightarrow x - 2y + 3z = -13$

21. $P(1, -1, 2)$, $Q(2, 1, 3)$ and $R(-1, 2, -1) \Rightarrow \vec{PQ} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\vec{PR} = -2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ and $\vec{PQ} \times \vec{PR}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ -2 & 3 & -3 \end{vmatrix} = -9\mathbf{i} + \mathbf{j} + 7\mathbf{k} \text{ is normal to the plane} \Rightarrow (-9)(x-1) + (1)(y+1) + (7)(z-2) = 0$$

$$\Rightarrow -9x + y + 7z = 4$$

22. $P(1, 0, 0)$, $Q(0, 1, 0)$ and $R(0, 0, 1) \Rightarrow \vec{PQ} = -\mathbf{i} + \mathbf{j}$, $\vec{PR} = -\mathbf{i} + \mathbf{k}$ and $\vec{PQ} \times \vec{PR}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \text{ is normal to the plane} \Rightarrow (1)(x-1) + (1)(y-0) + (1)(z-0) = 0$$

$$\Rightarrow x + y + z = 1$$

23. $(0, -\frac{1}{2}, -\frac{3}{2})$, since $t = -\frac{1}{2}$, $y = -\frac{1}{2}$ and $z = -\frac{3}{2}$ when $x = 0$; $(-1, 0, -3)$, since $t = -1$, $x = -1$ and $z = -3$

when $y = 0$; $(1, -1, 0)$, since $t = 0$, $x = 1$ and $y = -1$ when $z = 0$

24. $x = 2t$, $y = -t$, $z = -t$ represents a line containing the origin and perpendicular to the plane $2x - y - z = 4$; this line intersects the plane $3x - 5y + 2z = 6$ when t is the solution of $3(2t) - 5(-t) + 2(-t) = 6$

$$\Rightarrow t = \frac{2}{3} \Rightarrow \left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right) \text{ is the point of intersection}$$

25. $\mathbf{n}_1 = \mathbf{i}$ and $\mathbf{n}_2 = \mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k} \Rightarrow$ the desired angle is $\cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

26. The direction of the line is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$. Since the point $(-5, 3, 0)$ is on

both planes, the desired line is $x = -5 + 5t$, $y = 3 - t$, $z = -3t$.

27. The direction of the intersection is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 5 & -2 & -1 \end{vmatrix} = -6\mathbf{i} - 9\mathbf{j} - 12\mathbf{k} = -3(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$ and is the

same as the direction of the given line.

28. (a) The corresponding normals are $\mathbf{n}_1 = 3\mathbf{i} + 6\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and since $\mathbf{n}_1 \cdot \mathbf{n}_2 = (3)(2) + (0)(2) + (6)(-1) = 6 + 0 - 6 = 0$, we have that the planes are orthogonal

- (b) The line of intersection is parallel to $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 6 \\ 2 & 2 & -1 \end{vmatrix} = -12\mathbf{i} + 15\mathbf{j} + 6\mathbf{k}$. Now to find a point in

$$\text{the intersection, solve } \begin{cases} 3x + 6z = 1 \\ 2x + 2y - z = 3 \end{cases} \Rightarrow \begin{cases} 3x + 6z = 1 \\ 12x + 12y - 6z = 18 \end{cases} \Rightarrow 15x + 12y = 19 \Rightarrow x = 0 \text{ and } y = \frac{19}{12}$$

$$\Rightarrow \left(0, \frac{19}{12}, \frac{1}{6}\right) \text{ is a point on the line we seek. Therefore, the line is } x = -12t, y = \frac{19}{12} + 15t \text{ and } z = \frac{1}{6} + 6t.$$

29. A vector in the direction of the plane's normal is $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 7\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ and $P(1, 2, 3)$ on

$$\text{the plane} \Rightarrow 7(x-1) - 3(y-2) - 5(z-3) = 0 \Rightarrow 7x - 3y - 5z = -14.$$

30. $\mathbf{n} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$ is normal to the plane $\Rightarrow \mathbf{n} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 0\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} = -3\mathbf{j} + 3\mathbf{k}$ is orthogonal

to $\mathbf{v}$ and parallel to the plane

31. The vector $\mathbf{v} \times \mathbf{w}$ is normal to the plane of $\mathbf{v}$ and $\mathbf{w} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is orthogonal to $\mathbf{u}$ and parallel to the plane:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = -5\mathbf{i} + 3\mathbf{j} - \mathbf{k} \text{ and } \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ -5 & 3 & -1 \end{vmatrix} = -2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$$

$$\Rightarrow |\mathbf{u} \times (\mathbf{v} \times \mathbf{w})| = \sqrt{4 + 9 + 1} = \sqrt{14} \text{ and } \frac{1}{\sqrt{14}}(-2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \text{ is the desired unit vector.}$$

32. A vector parallel to the line of intersection is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

$$\Rightarrow |\mathbf{v}| = \sqrt{25 + 1 + 9} = \sqrt{35} \Rightarrow 2\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \frac{2}{\sqrt{35}}(5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) \text{ is the desired vector.}$$

33. The line containing $(0, 0, 0)$ normal to the plane is represented by $x = 2t$, $y = -t$, and $z = -t$. This line intersects the plane $3x - 5y + 2z = 6$ when $3(2t) - 5(-t) + 2(-t) = 6 \Rightarrow t = \frac{2}{3} \Rightarrow$ the point is $\left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$.

34. The line is represented by $x = 3 + 2t$, $y = 2 - t$, and $z = 1 + 2t$. It meets the plane $2x - y + 2z = -2$ when $2(3 + 2t) - (2 - t) + 2(1 + 2t) = -2 \Rightarrow t = -\frac{8}{9} \Rightarrow$ the point is $\left(\frac{11}{9}, \frac{26}{9}, -\frac{7}{9}\right)$.

35. The vector $\vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -3 & 0 & 1 \end{vmatrix} = -\mathbf{i} - 11\mathbf{j} - 3\mathbf{k}$ is normal to the plane.

- (a) No, the plane is not orthogonal to $\vec{PQ} \times \vec{PR}$.
 (b) No, these equations represent a line, not a plane.
 (c) No, the plane $(x + 2) + 11(y - 1) - 3z = 0$ has normal $\mathbf{i} + 11\mathbf{j} - 3\mathbf{k}$ which is not parallel to $\vec{PQ} \times \vec{PR}$.
 (d) No, this vector equation is equivalent to the equations $3y + 3z = 3$, $3x - 2z = -6$, and $3x + 2y = -4$
 $\Rightarrow x = -\frac{4}{3} - \frac{2}{3}t$, $y = t$, $z = 1 - t$, which represents a line, not a plane.
 (e) Yes, this is a plane containing the point $R(-2, 1, 0)$ with normal $\vec{PQ} \times \vec{PR}$.
36. (a) The line through A and B is $x = 1 + t$, $y = -t$, $z = -1 + 5t$; the line through C and D must be parallel and is L_1 : $x = 1 + t$, $y = 2 - t$, $z = 3 + 5t$. The line through B and C is $x = 1$, $y = 2 + 2s$, $z = 3 + 4s$; the line through A and D must be parallel and is L_2 : $x = 2$, $y = -1 + 2s$, $z = 4 + 4s$. The lines L_1 and L_2 intersect at $D(2, 1, 8)$ where $t = 1$ and $s = 1$.
- (b) $\cos \theta = \frac{(2\mathbf{j} + 4\mathbf{k}) \cdot (\mathbf{i} - \mathbf{j} + 5\mathbf{k})}{\sqrt{20} \sqrt{27}} = \frac{3}{\sqrt{15}}$
- (c) $\left(\frac{\vec{BA} \cdot \vec{BC}}{\vec{BC} \cdot \vec{BC}}\right) \vec{BC} = \frac{18}{20} \vec{BC} = \frac{9}{5}(\mathbf{j} + 2\mathbf{k})$ where $\vec{BA} = \mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and $\vec{BC} = 2\mathbf{j} + 4\mathbf{k}$
- (d) $\text{area} = |(2\mathbf{j} + 4\mathbf{k}) \times (\mathbf{i} - \mathbf{j} + 5\mathbf{k})| = |14\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}| = 6\sqrt{6}$

- (e) From part (d), $\mathbf{n} = 14\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ is normal to the plane $\Rightarrow 14(x-1) + 4(y-0) - 2(z+1) = 0$
 $\Rightarrow 7x + 2y - z = 8$.
- (f) From part (d), $\mathbf{n} = 14\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \Rightarrow$ the area of the projection on the yz -plane is $|\mathbf{n} \cdot \mathbf{i}| = 14$; the area of the projection on the xy -plane is $|\mathbf{n} \cdot \mathbf{j}| = 4$; and the area of the projection on the xz -plane is $|\mathbf{n} \cdot \mathbf{k}| = 2$.

37. The line L passes through the point $P(0, 0, -1)$ parallel to $\mathbf{v} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$. With $\vec{PS} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and

$$\vec{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ -1 & 1 & 1 \end{vmatrix} = (2-1)\mathbf{i} + (-1-2)\mathbf{j} + (2+2)\mathbf{k} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}, \text{ we find the distance}$$

$$d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}} = \frac{\sqrt{26}}{\sqrt{3}} = \frac{\sqrt{78}}{3}.$$

38. The line L passes through the point $P(2, 2, 0)$ parallel to $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. With $\vec{PS} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and

$$\vec{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = (2-1)\mathbf{i} + (1+2)\mathbf{j} + (-2-2)\mathbf{k} = \mathbf{i} + 3\mathbf{j} - 4\mathbf{k}, \text{ we find the distance}$$

$$d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}} = \frac{\sqrt{26}}{\sqrt{3}} = \frac{\sqrt{78}}{3}.$$

39. The point $P(4, 0, 0)$ lies on the plane $x - y = 4$, and $\vec{PS} = (6-4)\mathbf{i} + 0\mathbf{j} + (-6+0)\mathbf{k} = 2\mathbf{i} - 6\mathbf{k}$ with $\mathbf{n} = \mathbf{i} - \mathbf{j}$

$$\Rightarrow d = \frac{|\mathbf{n} \cdot \vec{PS}|}{|\mathbf{n}|} = \frac{|2+0+0|}{\sqrt{1+1+0}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

40. The point $P(0, 0, 2)$ lies on the plane $2x + 3y + z = 2$, and $\vec{PS} = (3-0)\mathbf{i} + (0-0)\mathbf{j} + (10+2)\mathbf{k} = 3\mathbf{i} + 8\mathbf{k}$ with

$$\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow d = \frac{|\mathbf{n} \cdot \vec{PS}|}{|\mathbf{n}|} = \frac{|6+0+8|}{\sqrt{4+9+1}} = \frac{14}{\sqrt{14}} = \sqrt{14}.$$

41. A normal to the plane is $\mathbf{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{vmatrix} = -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} \Rightarrow$ the distance is $d = \frac{|\vec{AP} \cdot \mathbf{n}|}{|\mathbf{n}|}$

$$= \frac{|(\mathbf{i} + 4\mathbf{j}) \cdot (-\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})|}{\sqrt{1+4+4}} = \frac{|-1-8+0|}{3} = 3$$

42. $P(0, 0, 0)$ lies on the plane $2x + 3y + 5z = 0$, and $\vec{PS} = 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ with $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} \Rightarrow$

$$d = \frac{|\mathbf{n} \cdot \vec{PS}|}{|\mathbf{n}|} = \frac{|4+6+15|}{\sqrt{4+9+25}} = \frac{25}{\sqrt{38}}$$

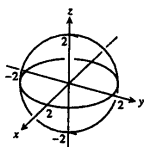
43. $\vec{AB} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\vec{CD} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$, and $\vec{AC} = 2\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 1 & 4 & -1 \end{vmatrix} = -5\mathbf{i} - \mathbf{j} - 9\mathbf{k} \Rightarrow$ the distance is

$$d = \left| \frac{(2\mathbf{i} + \mathbf{j}) \cdot (-5\mathbf{i} - \mathbf{j} - 9\mathbf{k})}{\sqrt{25 + 1 + 81}} \right| = \frac{11}{\sqrt{107}}$$

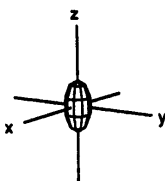
44. $\vec{AB} = -2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\vec{CD} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, and $\vec{AC} = -3\mathbf{i} + 3\mathbf{j} \Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 7\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} \Rightarrow$ the distance is

$$d = \left| \frac{(-3\mathbf{i} + 3\mathbf{j}) \cdot (7\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})}{\sqrt{49 + 9 + 4}} \right| = \frac{12}{\sqrt{62}}$$

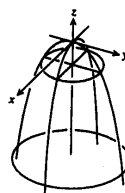
45. $x^2 + y^2 + z^2 = 4$



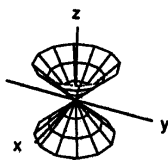
46. $4x^2 + 4y^2 + z^2 = 4$



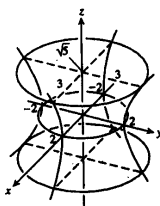
47. $z = -(x^2 + y^2)$



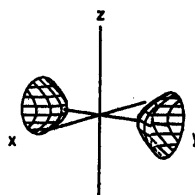
48. $x^2 + y^2 = z^2$



49. $x^2 + y^2 - z^2 = 4$



50. $y^2 - x^2 - z^2 = 1$



51. $\mathbf{r} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = (-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-2 \sin t)^2 + (2 \cos t)^2 + (2t)^2}$

$$= 2\sqrt{1 + t^2} \Rightarrow \text{Length} = \int_0^{\pi/4} 2\sqrt{1 + t^2} \, dt = \left[t\sqrt{1 + t^2} + \ln|t + \sqrt{1 + t^2}| \right]_0^{\pi/4} = \frac{\pi}{4}\sqrt{1 + \frac{\pi^2}{16}} + \ln\left(\frac{\pi}{4} + \sqrt{1 + \frac{\pi^2}{16}}\right)$$

$$52. \mathbf{r} = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + 2t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = (-3 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 3t^{1/2}\mathbf{k}$$

$$\Rightarrow |\mathbf{v}| = \sqrt{(-3 \sin t)^2 + (3 \cos t)^2 + (3t^{1/2})^2} = \sqrt{9 + 9t} = 3\sqrt{1+t} \Rightarrow \text{Length} = \int_0^3 3\sqrt{1+t} \, dt = \left[2(1+t)^{3/2} \right]_0^3 = 14$$

$$53. \mathbf{r} = \frac{4}{9}(1+t)^{3/2}\mathbf{i} + \frac{4}{9}(1-t)^{3/2}\mathbf{j} + \frac{1}{3}t\mathbf{k} \Rightarrow \mathbf{v} = \frac{2}{3}(1+t)^{1/2}\mathbf{i} - \frac{2}{3}(1-t)^{1/2}\mathbf{j} + \frac{1}{3}\mathbf{k}$$

$$\Rightarrow |\mathbf{v}| = \sqrt{\left[\frac{2}{3}(1+t)^{1/2}\right]^2 + \left[-\frac{2}{3}(1-t)^{1/2}\right]^2 + \left(\frac{1}{3}\right)^2} = 1 \Rightarrow \mathbf{T} = \frac{2}{3}(1+t)^{1/2}\mathbf{i} - \frac{2}{3}(1-t)^{1/2}\mathbf{j} + \frac{1}{3}\mathbf{k}$$

$$\Rightarrow \mathbf{T}(0) = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}; \frac{d\mathbf{T}}{dt} = \frac{1}{3}(1+t)^{-1/2}\mathbf{i} + \frac{1}{3}(1-t)^{-1/2}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(0) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} \Rightarrow \left| \frac{d\mathbf{T}}{dt}(0) \right| = \frac{\sqrt{2}}{3}$$

$$\Rightarrow \mathbf{N}(0) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}; \mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{vmatrix} = -\frac{1}{3\sqrt{2}}\mathbf{i} + \frac{1}{3\sqrt{2}}\mathbf{j} + \frac{4}{3\sqrt{2}}\mathbf{k};$$

$$\mathbf{a} = \frac{1}{3}(1+t)^{-1/2}\mathbf{i} + \frac{1}{3}(1-t)^{-1/2}\mathbf{j} \Rightarrow \mathbf{a}(0) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} \text{ and } \mathbf{v}(0) = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \Rightarrow \mathbf{v}(0) \times \mathbf{a}(0)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{vmatrix} = -\frac{1}{9}\mathbf{i} + \frac{1}{9}\mathbf{j} + \frac{4}{9}\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = \frac{\sqrt{2}}{3} \Rightarrow \kappa(0) = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{\left(\frac{\sqrt{2}}{3}\right)}{1^3} = \frac{\sqrt{2}}{3};$$

$$\dot{\mathbf{a}} = -\frac{1}{6}(1+t)^{-3/2}\mathbf{i} + \frac{1}{6}(1-t)^{-3/2}\mathbf{j} \Rightarrow \dot{\mathbf{a}}(0) = -\frac{1}{6}\mathbf{i} + \frac{1}{6}\mathbf{j} \Rightarrow \tau(0) = \frac{\begin{vmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{6} & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{\left(\frac{1}{3}\right)\left(\frac{2}{18}\right)}{\left(\frac{\sqrt{2}}{3}\right)^2} = \frac{1}{6};$$

$$t = 0 \Rightarrow \left(\frac{4}{9}, \frac{4}{9}, 0\right) \text{ is the point on the curve}$$

$$54. \mathbf{r} = (e^t \sin 2t)\mathbf{i} + (e^t \cos 2t)\mathbf{j} + 2e^t\mathbf{k} \Rightarrow \mathbf{v} = (e^t \sin 2t + 2e^t \cos 2t)\mathbf{i} + (e^t \cos 2t - 2e^t \sin 2t)\mathbf{j} + 2e^t\mathbf{k}$$

$$\Rightarrow |\mathbf{v}| = \sqrt{(e^t \sin 2t + 2e^t \cos 2t)^2 + (e^t \cos 2t - 2e^t \sin 2t)^2 + (2e^t)^2} = 3e^t \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

$$= \left(\frac{1}{3} \sin 2t + \frac{2}{3} \cos 2t\right)\mathbf{i} + \left(\frac{1}{3} \cos 2t - \frac{2}{3} \sin 2t\right)\mathbf{j} + \frac{2}{3}\mathbf{k} \Rightarrow \mathbf{T}(0) = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k};$$

$$\frac{d\mathbf{T}}{dt} = \left(\frac{2}{3} \cos 2t - \frac{4}{3} \sin 2t\right)\mathbf{i} + \left(-\frac{2}{3} \sin 2t - \frac{4}{3} \cos 2t\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(0) = \frac{2}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} \Rightarrow \left| \frac{d\mathbf{T}}{dt}(0) \right| = \frac{2}{3}\sqrt{5}$$

$$\Rightarrow \mathbf{N}(0) = \frac{\left(\frac{2}{3}\mathbf{i} - \frac{4}{3}\mathbf{j}\right)}{\left(\frac{2\sqrt{5}}{3}\right)} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}; \mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \end{vmatrix} = \frac{4}{3\sqrt{5}}\mathbf{i} + \frac{2}{3\sqrt{5}}\mathbf{j} - \frac{5}{3\sqrt{5}}\mathbf{k};$$

$$\mathbf{a} = (4e^t \cos 2t - 3e^t \sin 2t)\mathbf{i} + (-3e^t \cos 2t - 4e^t \sin 2t)\mathbf{j} + 2e^t\mathbf{k} \Rightarrow \mathbf{a}(0) = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{v}(0) = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

$$\Rightarrow \mathbf{v}(0) \times \mathbf{a}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 4 & -3 & 2 \end{vmatrix} = 8\mathbf{i} + 4\mathbf{j} - 10\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = \sqrt{64 + 16 + 100} = 6\sqrt{5} \text{ and } |\mathbf{v}(0)| = 3$$

$$\Rightarrow \kappa(0) = \frac{6\sqrt{5}}{3^3} = \frac{2\sqrt{5}}{9};$$

$$\dot{\mathbf{a}} = (4e^t \cos 2t - 8e^t \sin 2t - 3e^t \sin 2t - 6e^t \cos 2t)\mathbf{i} + (-3e^t \cos 2t + 6e^t \sin 2t - 4e^t \sin 2t - 8e^t \cos 2t)\mathbf{j} + 2e^t\mathbf{k}$$

$$= (-2e^t \cos 2t - 11e^t \sin 2t)\mathbf{i} + (-11e^t \cos 2t + 2e^t \sin 2t)\mathbf{j} + 2e^t\mathbf{k} \Rightarrow \dot{\mathbf{a}}(0) = -2\mathbf{i} - 11\mathbf{j} + 2\mathbf{k}$$

$$\Rightarrow \tau(0) = \frac{\begin{vmatrix} 2 & 1 & 2 \\ 4 & -3 & 2 \\ -2 & -11 & 2 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{-80}{180} = -\frac{4}{9}; t = 0 \Rightarrow (0, 1, 2) \text{ is on the curve}$$

$$55. \mathbf{r} = t\mathbf{i} + \frac{1}{2}e^{2t}\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + e^{2t}\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1 + e^{4t}} \Rightarrow \mathbf{T} = \frac{1}{\sqrt{1 + e^{4t}}}\mathbf{i} + \frac{e^{2t}}{\sqrt{1 + e^{4t}}}\mathbf{j} \Rightarrow \mathbf{T}(\ln 2) = \frac{1}{\sqrt{17}}\mathbf{i} + \frac{4}{\sqrt{17}}\mathbf{j};$$

$$\frac{d\mathbf{T}}{dt} = \frac{-2e^{4t}}{(1 + e^{4t})^{3/2}}\mathbf{i} + \frac{2e^{2t}}{(1 + e^{4t})^{3/2}}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(\ln 2) = \frac{-32}{17\sqrt{17}}\mathbf{i} + \frac{8}{17\sqrt{17}}\mathbf{j} \Rightarrow \mathbf{N}(\ln 2) = -\frac{4}{\sqrt{17}}\mathbf{i} + \frac{1}{\sqrt{17}}\mathbf{j};$$

$$\mathbf{B}(\ln 2) = \mathbf{T}(\ln 2) \times \mathbf{N}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} & 0 \\ -\frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 \end{vmatrix} = \mathbf{k}; \mathbf{a} = 2e^{2t}\mathbf{j} \Rightarrow \mathbf{a}(\ln 2) = 8\mathbf{j} \text{ and } \mathbf{v}(\ln 2) = \mathbf{i} + 4\mathbf{j}$$

$$\Rightarrow \mathbf{v}(\ln 2) \times \mathbf{a}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 0 \\ 0 & 8 & 0 \end{vmatrix} = 8\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = 8 \text{ and } |\mathbf{v}(\ln 2)| = \sqrt{17} \Rightarrow \kappa(\ln 2) = \frac{8}{17\sqrt{17}}; \dot{\mathbf{a}} = 4e^{2t}\mathbf{j}$$

$$\Rightarrow \dot{\mathbf{a}}(\ln 2) = 16\mathbf{j} \Rightarrow \tau(\ln 2) = \frac{\begin{vmatrix} 1 & 4 & 0 \\ 0 & 8 & 0 \\ 0 & 16 & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = 0; t = \ln 2 \Rightarrow (\ln 2, 2, 0) \text{ is on the curve}$$

$$56. \mathbf{r} = (3 \cosh 2t)\mathbf{i} + (3 \sinh 2t)\mathbf{j} + 6t\mathbf{k} \Rightarrow \mathbf{v} = (6 \sinh 2t)\mathbf{i} + (6 \cosh 2t)\mathbf{j} + 6\mathbf{k}$$

$$\Rightarrow |\mathbf{v}| = \sqrt{36 \sinh^2 2t + 36 \cosh^2 2t + 36} = 6\sqrt{2} \cosh 2t \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}} \tanh 2t\right)\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}} \operatorname{sech} 2t\right)\mathbf{k}$$

$$\Rightarrow \mathbf{T}(\ln 2) = \frac{15}{17\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{8}{17\sqrt{2}}\mathbf{k}; \frac{d\mathbf{T}}{dt} = \left(\frac{2}{\sqrt{2}} \operatorname{sech}^2 2t\right)\mathbf{i} - \left(\frac{2}{\sqrt{2}} \operatorname{sech} 2t \tanh 2t\right)\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt}(\ln 2)$$

$$= \left(\frac{2}{\sqrt{2}}\right)\left(\frac{8}{17}\right)^2\mathbf{i} - \left(\frac{2}{\sqrt{2}}\right)\left(\frac{8}{17}\right)\left(\frac{15}{17}\right)\mathbf{k} = \frac{128}{289\sqrt{2}}\mathbf{i} - \frac{240}{289\sqrt{2}}\mathbf{k} \Rightarrow \left|\frac{d\mathbf{T}}{dt}(\ln 2)\right| = \sqrt{\left(\frac{128}{289\sqrt{2}}\right)^2 + \left(-\frac{240}{289\sqrt{2}}\right)^2} = \frac{8\sqrt{2}}{17}$$

$$\Rightarrow \mathbf{N}(\ln 2) = \frac{8}{17}\mathbf{i} - \frac{15}{17}\mathbf{k}; \mathbf{B}(\ln 2) = \mathbf{T}(\ln 2) \times \mathbf{N}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{15}{17\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{8}{17\sqrt{2}} \\ \frac{8}{17} & 0 & -\frac{15}{17} \end{vmatrix} = -\frac{15}{17\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} - \frac{8}{17\sqrt{2}}\mathbf{k};$$

$$\mathbf{a} = (12 \cosh 2t)\mathbf{i} + (12 \sinh 2t)\mathbf{j} \Rightarrow \mathbf{a}(\ln 2) = 12\left(\frac{17}{8}\right)\mathbf{i} + 12\left(\frac{15}{8}\right)\mathbf{j} = \frac{51}{2}\mathbf{i} + \frac{45}{2}\mathbf{j} \text{ and}$$

$$\mathbf{v}(\ln 2) = 6\left(\frac{15}{8}\right)\mathbf{i} + 6\left(\frac{17}{8}\right)\mathbf{j} + 6\mathbf{k} = \frac{45}{4}\mathbf{i} + \frac{51}{4}\mathbf{j} + 6\mathbf{k} \Rightarrow \mathbf{v}(\ln 2) \times \mathbf{a}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{45}{4} & \frac{51}{4} & 6 \\ \frac{51}{2} & \frac{45}{2} & 0 \end{vmatrix}$$

$$= -135\mathbf{i} + 153\mathbf{j} - 72\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = 153\sqrt{2} \text{ and } |\mathbf{v}(\ln 2)| = \frac{51}{4}\sqrt{2} \Rightarrow \kappa(\ln 2) = \frac{153\sqrt{2}}{\left(\frac{51}{4}\sqrt{2}\right)^3} = \frac{32}{867};$$

$$\dot{\mathbf{a}} = (24 \sinh 2t)\mathbf{i} + (24 \cosh 2t)\mathbf{j} \Rightarrow \dot{\mathbf{a}}(\ln 2) = 45\mathbf{i} + 51\mathbf{j} \Rightarrow \tau(\ln 2) = \frac{\begin{vmatrix} \frac{45}{4} & \frac{51}{4} & 6 \\ \frac{51}{2} & \frac{45}{2} & 0 \\ 45 & 51 & 0 \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{32}{867};$$

$$t = \ln 2 \Rightarrow \left(\frac{51}{8}, \frac{45}{8}, 6 \ln 2\right) \text{ is on the curve}$$

$$57. \mathbf{r} = (2 + 3t + 3t^2)\mathbf{i} + (4t + 4t^2)\mathbf{j} - (6 \cos t)\mathbf{k} \Rightarrow \mathbf{v} = (3 + 6t)\mathbf{i} + (4 + 8t)\mathbf{j} + (6 \sin t)\mathbf{k}$$

$$\Rightarrow |\mathbf{v}| = \sqrt{(3 + 6t)^2 + (4 + 8t)^2 + (6 \sin t)^2} = \sqrt{25 + 100t + 100t^2 + 36 \sin^2 t}$$

$$\Rightarrow \frac{d|\mathbf{v}|}{dt} = \frac{1}{2}(25 + 100t + 100t^2 + 36 \sin^2 t)^{-1/2}(100 + 200t + 72 \sin t \cos t) \Rightarrow \mathbf{a}_T(0) = \frac{d|\mathbf{v}|}{dt}(0) = 10;$$

$$\mathbf{a} = 6\mathbf{i} + 8\mathbf{j} + (t \cos t)\mathbf{k} \Rightarrow |\mathbf{a}| = \sqrt{6^2 + 8^2 + (6 \cos t)^2} = \sqrt{100 + 36 \cos^2 t} \Rightarrow |\mathbf{a}(0)| = \sqrt{136}$$

$$\Rightarrow \mathbf{a}_N = \sqrt{|\mathbf{a}|^2 - \mathbf{a}_T^2} = \sqrt{136 - 10^2} = \sqrt{36} = 6 \Rightarrow \mathbf{a}(0) = 10\mathbf{T} + 6\mathbf{N}$$

$$\begin{aligned}
 58. \quad \mathbf{r} &= (2+t)\mathbf{i} + (t+2t^2)\mathbf{j} + (1+t^2)\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + (1+4t)\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (1+4t)^2 + (2t)^2} \\
 &= \sqrt{2+8t+20t^2} \Rightarrow \frac{d|\mathbf{v}|}{dt} = \frac{1}{2}(2+8t+20t^2)^{-1/2}(8+40t) \Rightarrow \mathbf{a}_T = \frac{d|\mathbf{v}|}{dt}(0) = 2\sqrt{2}; \mathbf{a} = 4\mathbf{j} + 2\mathbf{k} \\
 &\Rightarrow |\mathbf{a}| = \sqrt{4^2 + 2^2} = \sqrt{20} \Rightarrow \mathbf{a}_N = \sqrt{|\mathbf{a}|^2 - \mathbf{a}_T^2} = \sqrt{20 - (2\sqrt{2})^2} = \sqrt{12} = 2\sqrt{3} \Rightarrow \mathbf{a}(0) = 2\sqrt{2}\mathbf{T} + 2\sqrt{3}\mathbf{N}
 \end{aligned}$$

$$\begin{aligned}
 59. \quad \mathbf{r} &= (\sin t)\mathbf{i} + (\sqrt{2} \cos t)\mathbf{j} + (\sin t)\mathbf{k} \Rightarrow \mathbf{v} = (\cos t)\mathbf{i} - (\sqrt{2} \sin t)\mathbf{j} + (\cos t)\mathbf{k} \\
 &\Rightarrow |\mathbf{v}| = \sqrt{(\cos t)^2 + (-\sqrt{2} \sin t)^2 + (\cos t)^2} = \sqrt{2} \Rightarrow T = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}} \cos t\right)\mathbf{i} - (\sin t)\mathbf{j} + \left(\frac{1}{\sqrt{2}} \cos t\right)\mathbf{k}; \\
 \frac{d\mathbf{T}}{dt} &= \left(-\frac{1}{\sqrt{2}} \sin t\right)\mathbf{i} - (\cos t)\mathbf{j} - \left(\frac{1}{\sqrt{2}} \sin t\right)\mathbf{k} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(-\frac{1}{\sqrt{2}} \sin t\right)^2 + (-\cos t)^2 + \left(-\frac{1}{\sqrt{2}} \sin t\right)^2} = 1
 \end{aligned}$$

$$\Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(-\frac{1}{\sqrt{2}} \sin t\right)\mathbf{i} - (\cos t)\mathbf{j} - \left(\frac{1}{\sqrt{2}} \sin t\right)\mathbf{k}; \mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}} \cos t & -\sin t & \frac{1}{\sqrt{2}} \cos t \\ -\frac{1}{\sqrt{2}} \sin t & -\cos t & -\frac{1}{\sqrt{2}} \sin t \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{k}; \mathbf{a} = (-\sin t)\mathbf{i} - (\sqrt{2} \cos t)\mathbf{j} - (\sin t)\mathbf{k} \Rightarrow \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & -\sqrt{2} \sin t & \cos t \\ -\sin t & -\sqrt{2} \cos t & -\sin t \end{vmatrix}$$

$$= \sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = \sqrt{4} = 2 \Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{2}{(\sqrt{2})^3} = \frac{1}{\sqrt{2}}; \dot{\mathbf{a}} = (-\cos t)\mathbf{i} + (\sqrt{2} \sin t)\mathbf{j} - (\cos t)\mathbf{k}$$

$$\Rightarrow \tau = \frac{\begin{vmatrix} \cos t & -\sqrt{2} \sin t & \cos t \\ -\sin t & -\sqrt{2} \cos t & -\sin t \\ -\cos t & \sqrt{2} \sin t & -\cos t \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{(\cos t)(\sqrt{2}) - (\sqrt{2} \sin t)(0) + (\cos t)(-\sqrt{2})}{4} = 0$$

$$\begin{aligned}
 60. \quad \mathbf{r} &= \mathbf{i} + (5 \cos t)\mathbf{j} + (3 \sin t)\mathbf{k} \Rightarrow \mathbf{v} = (-5 \sin t)\mathbf{j} + (3 \cos t)\mathbf{k} \Rightarrow \mathbf{a} = (-5 \cos t)\mathbf{j} - (3 \sin t)\mathbf{k} \\
 &\Rightarrow \mathbf{v} \cdot \mathbf{a} = 25 \sin t \cos t - 9 \sin t \cos t = 16 \sin t \cos t; \mathbf{v} \cdot \mathbf{a} = 0 \Rightarrow 16 \sin t \cos t = 0 \Rightarrow \sin t = 0 \text{ or } \cos t = 0 \\
 &\Rightarrow t = 0, \frac{\pi}{2} \text{ or } \pi
 \end{aligned}$$

$$\begin{aligned}
 61. \quad \mathbf{r} &= 2\mathbf{i} + \left(4 \sin \frac{t}{2}\right)\mathbf{j} + \left(3 - \frac{t}{\pi}\right)\mathbf{k} \Rightarrow 0 = \mathbf{r} \cdot (\mathbf{i} - \mathbf{j}) = 2(1) + \left(4 \sin \frac{t}{2}\right)(-1) \Rightarrow 0 = 2 - 4 \sin \frac{t}{2} \Rightarrow \sin \frac{t}{2} = \frac{1}{2} \Rightarrow \frac{t}{2} = \frac{\pi}{6} \\
 &\Rightarrow t = \frac{\pi}{3} \text{ (for the first time)}
 \end{aligned}$$

$$62. \quad \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1 + 4t^2 + 9t^4} \Rightarrow |\mathbf{v}(1)| = \sqrt{14}$$

$$\Rightarrow \mathbf{T}(1) = \frac{1}{\sqrt{14}}\mathbf{i} + \frac{2}{\sqrt{14}}\mathbf{j} + \frac{3}{\sqrt{14}}\mathbf{k}, \text{ which is normal to the normal plane}$$

$\Rightarrow \frac{1}{\sqrt{14}}(x-1) + \frac{2}{\sqrt{14}}(y-1) + \frac{3}{\sqrt{14}}(z-1) = 0$ or $x+2y+3z=6$ is an equation of the normal plane. Next we calculate $\mathbf{N}(1)$ which is normal to the rectifying plane. Now, $\mathbf{a} = 2\mathbf{j} + 6\mathbf{k} \Rightarrow \mathbf{a}(1) = 2\mathbf{j} + 6\mathbf{k} \Rightarrow \mathbf{v}(1) \times \mathbf{a}(1)$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} = 6\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} \Rightarrow |\mathbf{v}(1) \times \mathbf{a}(1)| = \sqrt{76} \Rightarrow \kappa(1) = \frac{\sqrt{76}}{(\sqrt{14})^3} = \frac{\sqrt{19}}{7\sqrt{14}}; \frac{ds}{dt} = |\mathbf{v}(t)| \Rightarrow \frac{d^2s}{dt^2} \Big|_{t=1}$$

$$= \frac{1}{2}(1+4t^2+9t^4)^{-1/2}(8t+36t^3) \Big|_{t=1} = \frac{22}{\sqrt{14}}, \text{ so } \mathbf{a} = \frac{d^2s}{dt^2} \mathbf{T} + \kappa \left(\frac{ds}{dt}\right)^2 \mathbf{N} \Rightarrow 2\mathbf{j} + 6\mathbf{k}$$

$$= \frac{22}{\sqrt{14}} \left(\frac{\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}}{\sqrt{14}} \right) + \frac{\sqrt{19}}{7\sqrt{14}} (\sqrt{14})^2 \mathbf{N} \Rightarrow \mathbf{N} = \frac{\sqrt{14}}{2\sqrt{19}} \left(-\frac{11}{7}\mathbf{i} - \frac{8}{7}\mathbf{j} + \frac{9}{7}\mathbf{k} \right) \Rightarrow -\frac{11}{7}(x-1) - \frac{8}{7}(y-1) + \frac{9}{7}(z-1)$$

$= 0$ or $11x+8y-9z=10$ is an equation of the rectifying plane. Finally, $\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1)$

$$= \left(\frac{\sqrt{14}}{2\sqrt{19}} \right) \left(\frac{1}{\sqrt{14}} \right) \left(\frac{1}{7} \right) \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -11 & -8 & 9 \end{vmatrix} = \frac{1}{\sqrt{19}}(3\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \Rightarrow 3(x-1) - 3(y-1) + (z-1) = 0 \text{ or } 3x - 3y + z$$

$= 1$ is an equation of the osculating plane.

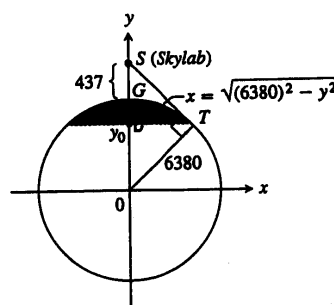
63. $\mathbf{r} = e^t\mathbf{i} + (\sin t)\mathbf{j} + \ln(1-t)\mathbf{k} \Rightarrow \mathbf{v} = e^t\mathbf{i} + (\cos t)\mathbf{j} - \left(\frac{1}{1-t}\right)\mathbf{k} \Rightarrow \mathbf{v}(0) = \mathbf{i} + \mathbf{j} - \mathbf{k}; \mathbf{r}(0) = \mathbf{i} \Rightarrow (1, 0, 0)$ is on the line
 $\Rightarrow x = 1 + t, y = t, \text{ and } z = -t$ are parametric equations of the line

64. $\mathbf{r} = (\sqrt{2} \cos t)\mathbf{i} + (\sqrt{2} \sin t)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = (-\sqrt{2} \sin t)\mathbf{i} + (\sqrt{2} \cos t)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{4}\right)$
 $= (-\sqrt{2} \sin \frac{\pi}{4})\mathbf{i} + (\sqrt{2} \cos \frac{\pi}{4})\mathbf{j} + \mathbf{k} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ is a vector tangent to the helix when $t = \frac{\pi}{4} \Rightarrow$ the tangent line
 is parallel to $\mathbf{v}\left(\frac{\pi}{4}\right)$; also $\mathbf{r}\left(\frac{\pi}{4}\right) = (\sqrt{2} \cos \frac{\pi}{4})\mathbf{i} + (\sqrt{2} \sin \frac{\pi}{4})\mathbf{j} + \frac{\pi}{4}\mathbf{k} \Rightarrow$ the point $(1, 1, \frac{\pi}{4})$ is on the line
 $\Rightarrow x = 1 - t, y = 1 + t, \text{ and } z = \frac{\pi}{4} + t$ are parametric equations of the line

65. $\Delta SOT \approx \Delta TOD \Rightarrow \frac{DO}{OT} = \frac{OT}{SO} \Rightarrow \frac{y_0}{6380} = \frac{6380}{6380 + 437}$

$$\Rightarrow y_0 = \frac{6380^2}{6817} \Rightarrow y_0 \approx 5971 \text{ km};$$

$$\begin{aligned} VA &= \int_{5971}^{6380} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \\ &= 2\pi \int_{5971}^{6380} \sqrt{6380^2 - y^2} \left(\frac{6380}{\sqrt{6380^2 - y^2}} \right) dy \\ &= 2\pi \int_{5971}^{6380} 6380 dy = 2\pi [6380y]_{5971}^{6380} \end{aligned}$$



$$= 16,395,469 \text{ km}^2 \approx 1.639 \times 10^7 \text{ km}^2;$$

$$\text{percentage visible} \approx \frac{16,395,469 \text{ km}^2}{4\pi(6380 \text{ km})^2} \approx 3.21\%$$

$$\begin{aligned} 66. \quad \ddot{s} &= \frac{d}{dt} \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{x}\ddot{x} + \dot{y}\ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Rightarrow \dot{x}^2 + \dot{y}^2 - \ddot{s}^2 = \dot{x}^2 + \dot{y}^2 - \frac{(\dot{x}\ddot{x} + \dot{y}\ddot{y})^2}{\dot{x}^2 + \dot{y}^2} \\ &= \frac{(\dot{x}^2 + \dot{y}^2)(\dot{x}^2 + \dot{y}^2) - (\dot{x}^2\ddot{x}^2 + 2\dot{x}\ddot{x}\dot{y}\ddot{y} + \dot{y}^2\ddot{y}^2)}{\dot{x}^2 + \dot{y}^2} = \frac{\dot{x}^2\ddot{y}^2 + \dot{y}^2\ddot{x}^2 - 2\dot{x}\ddot{x}\dot{y}\ddot{y}}{\dot{x}^2 + \dot{y}^2} = \frac{(\dot{x}\ddot{y} - \dot{y}\ddot{x})^2}{\dot{x}^2 + \dot{y}^2} \\ &\Rightarrow \sqrt{\dot{x}^2 + \dot{y}^2 - \ddot{s}^2} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Rightarrow \frac{\dot{x}^2 + \dot{y}^2}{\sqrt{\dot{x}^2 + \dot{y}^2 - \ddot{s}^2}} = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|} = \frac{1}{\kappa} = \rho \end{aligned}$$

CHAPTER 10 ADDITIONAL EXERCISES—THEORY, EXAMPLES, APPLICATIONS

- Information from ship A indicates the submarine is now on the line L_1 : $x = 4 + 2t$, $y = 3t$, $z = -\frac{1}{3}t$; information from ship B indicates the submarine is now on the line L_2 : $x = 18s$, $y = 5 - 6s$, $z = -s$. The current position of the sub is $(6, 3, -\frac{1}{3})$ and occurs when the lines intersect at $t = 1$ and $s = \frac{1}{3}$. The straight line path of the submarine contains both points $P(2, -1, -\frac{1}{3})$ and $Q(6, 3, -\frac{1}{3})$; the line representing this path is L : $x = 2 + 4t$, $y = -1 + 4t$, $z = -\frac{1}{3}$. The submarine traveled the distance between P and Q in 4 minutes $\Rightarrow$ a speed of $\frac{|\vec{PQ}|}{4} = \frac{\sqrt{32}}{4} = \sqrt{2}$ thousand ft/min. In 20 minutes the submarine will move $20\sqrt{2}$ thousand ft from Q along the line $L \Rightarrow 20\sqrt{2} = \sqrt{(2 + 4t - 6)^2 + (-1 + 4t - 3)^2 + 0^2} \Rightarrow 800 = 16(t - 1)^2 + 16(t - 1)^2 = 32(t - 1)^2 \Rightarrow (t - 1)^2 = \frac{800}{32} = 25 \Rightarrow t = 6 \Rightarrow$ the submarine will be located at $(26, 23, -\frac{1}{3})$ in 20 minutes.
- H_2 stops its flight when $6 + 110t = 446 \Rightarrow t = 4$ hours. After 6 hours, H_1 is at $P(246, 57, 9)$ while H_2 is at $(446, 13, 0)$. The distance between P and Q is $\sqrt{(246 - 446)^2 + (57 - 13)^2 + (9 - 0)^2} \approx 204.98$ miles. At 150 mph, it would take about 1.37 hours for H_1 to reach H_2 .
- Torque $= |\vec{PQ} \times \mathbf{F}| \Rightarrow 15 \text{ ft} \cdot \text{lb} = |\vec{PQ}| |\mathbf{F}| \sin \frac{\pi}{2} = \frac{3}{4} \text{ ft} \cdot |\mathbf{F}| \Rightarrow |\mathbf{F}| = 20 \text{ lb}$
- Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ be the vector from O to A and $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ be the vector from O to B. The vector $\mathbf{v}$ orthogonal to $\mathbf{a}$ and $\mathbf{b} \Rightarrow \mathbf{v}$ is parallel to $\mathbf{b} \times \mathbf{a}$ (since the rotation is clockwise). Now $\mathbf{b} \times \mathbf{a} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$; $\text{proj}_{\mathbf{a}} \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \right) \mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow (2, 2, 2)$ is the center of the circular path $(1, 3, 2)$ takes $\Rightarrow$ radius $= \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2} \Rightarrow$ arc length per second covered by the point is $\frac{3}{2}\sqrt{2}$ units/sec $= |\mathbf{v}|$ (velocity is constant). A unit vector in the direction of $\mathbf{v}$ is $\frac{\mathbf{b} \times \mathbf{a}}{|\mathbf{b} \times \mathbf{a}|} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} - \frac{2}{\sqrt{6}}\mathbf{k} \Rightarrow \mathbf{v} = |\mathbf{v}| \left(\frac{\mathbf{b} \times \mathbf{a}}{|\mathbf{b} \times \mathbf{a}|} \right) = \frac{3}{2}\sqrt{2} \left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} - \frac{2}{\sqrt{6}}\mathbf{k} \right) = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} - \sqrt{3}\mathbf{k}$

5. (a) If $P(x, y, z)$ is a point in the plane determined by the three points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ and $P_3(x_3, y_3, z_3)$, then the vectors $\overrightarrow{PP_1}$, $\overrightarrow{PP_2}$ and $\overrightarrow{PP_3}$ all lie in the plane. Thus $\overrightarrow{PP_1} \cdot (\overrightarrow{PP_2} \times \overrightarrow{PP_3}) = 0$

$$\Rightarrow \begin{vmatrix} x_1 - x & y_1 - y & z_1 - z \\ x_2 - x & y_2 - y & z_2 - z \\ x_3 - x & y_3 - y & z_3 - z \end{vmatrix} = 0 \text{ by the determinant formula for the triple scalar product in Section 10.2.}$$

- (b) Subtract row 1 from rows 2, 3, and 4 and evaluate the resulting determinant (which has the same value as the given determinant) by cofactor expansion about column 4. This expansion is exactly the determinant in part (a) so we have all points $P(x, y, z)$ in the plane determined by $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$.

6. Let $L_1: x = a_1s + b_1, y = a_2s + b_2, z = a_3s + b_3$ and $L_2: x = c_1t + d_1, y = c_2t + d_2, z = c_3t + d_3$. If $L_1 \parallel L_2$,

$$\text{then for some } k, a_i = kc_i, i = 1, 2, 3 \text{ and the determinant } \begin{vmatrix} a_1 & c_1 & b_1 - d_1 \\ a_2 & c_2 & b_2 - d_2 \\ a_3 & c_3 & b_3 - d_3 \end{vmatrix} = \begin{vmatrix} kc_1 & c_1 & b_1 - d_1 \\ kc_2 & c_2 & b_2 - d_2 \\ kc_3 & c_3 & b_3 - d_3 \end{vmatrix} = 0,$$

since the first column is a multiple of the second column. The lines L_1 and L_2 intersect if and only if the

$$\text{system } \begin{cases} a_1s - c_1t + (b_1 - d_1) = 0 \\ a_2s - c_2t + (b_2 - d_2) = 0 \\ a_3s - c_3t + (b_3 - d_3) = 0 \end{cases} \text{ has a nontrivial solution } \Leftrightarrow \text{the determinant of the coefficients is zero.}$$

7. Let (x, y, z) be any point on $Ax + By + Cz - D = 0$. Let $\overrightarrow{QP_1} = (x - x_1)\mathbf{i} + (y - y_1)\mathbf{j} + (z - z_1)\mathbf{k}$, and

$$\mathbf{n} = \frac{A\mathbf{i} + B\mathbf{j} + C\mathbf{k}}{\sqrt{A^2 + B^2 + C^2}}. \text{ The distance is } |\text{proj}_{\mathbf{n}} \overrightarrow{QP_1}| = \left| ((x - x_1)\mathbf{i} + (y - y_1)\mathbf{j} + (z - z_1)\mathbf{k}) \cdot \left(\frac{A\mathbf{i} + B\mathbf{j} + C\mathbf{k}}{\sqrt{A^2 + B^2 + C^2}} \right) \right|$$

$$= \frac{|Ax_1 + By_1 + Cz_1 - (Ax + By + Cz)|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|Ax_1 + By_1 + Cz_1 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

8. Since both tangent planes are parallel, one-half of the distance between them is equal to the radius of the

sphere, i.e., $r = \frac{1}{2} \frac{|3 - 9|}{\sqrt{1 + 1 + 1}} = \sqrt{3}$ (see also Exercise 9a). Clearly, the points $(1, 2, 3)$ and $(-1, -2, -3)$

are on the line containing the sphere's center. Hence, the line containing the center is $x = 1 + 2t$,

$y = 2 + 4t, z = 3 + 6t$. The distance from the plane $x + y + z - 3 = 0$ to the center is $\sqrt{3}$

$$\Rightarrow \frac{|(1 + 2t) + (2 + 4t) + (3 + 6t) - 3|}{\sqrt{1 + 1 + 1}} = \sqrt{3} \text{ from part (a)} \Rightarrow t = 0 \Rightarrow \text{the center is at } (1, 2, 3). \text{ Therefore}$$

an equation of the sphere is $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 3$.

9. (a) If (x_1, y_1, z_1) is on the plane $Ax + By + Cz = D_1$, then the distance d between the planes is

$$d = \frac{|Ax_1 + By_1 + Cz_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|D_1 - D_2|}{|A\mathbf{i} + B\mathbf{j} + C\mathbf{k}|}, \text{ since } Ax_1 + By_1 + Cz_1 = D_1, \text{ by Exercise 7.}$$

$$(b) d = \frac{|12 - 6|}{\sqrt{4 + 9 + 1}} = \frac{6}{\sqrt{14}}$$

$$10. \frac{|2(3) + (-1)(2) + 2(-1) + 4|}{\sqrt{14}} = \frac{|2(3) + (-1)(2) + 2(-1) + D|}{\sqrt{14}} \Rightarrow D = -8 \text{ or } 4 \Rightarrow \text{the desired plane is } 2x - y + 2z = 8$$

11. Let $\mathbf{n} = \vec{AB} \times \vec{BC}$ and $P(x, y, z)$ be any point in the plane determined by A, B and C. Then the point D lies in this plane if and only if $\vec{AD} \cdot \mathbf{n} = 0 \Leftrightarrow \vec{AD} \cdot (\vec{AB} \times \vec{BC}) = 0$.

$$12. (a) \mathbf{u} \times \mathbf{v} = 4\mathbf{i} \times \mathbf{j} = 4\mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0}; (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = 0\mathbf{v} - 0\mathbf{u} = \mathbf{0}; \mathbf{v} \times \mathbf{w} = 4\mathbf{i} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0};$$

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = 0\mathbf{v} - 0\mathbf{w} = \mathbf{0}$$

$$(b) \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 3 \\ -1 & 2 & -1 \end{vmatrix} = -10\mathbf{i} - 2\mathbf{j} + 6\mathbf{k};$$

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = -4(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - 2(\mathbf{i} - \mathbf{j} + \mathbf{k}) = -10\mathbf{i} - 2\mathbf{j} + 6\mathbf{k};$$

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 3 & 4 & 5 \end{vmatrix} = -9\mathbf{i} - 2\mathbf{j} + 7\mathbf{k};$$

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = -4(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - (-1)(-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -9\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$$

$$(c) \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -4 \\ 1 & 0 & 2 \end{vmatrix} = -4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k};$$

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = 2(2\mathbf{i} - \mathbf{j} + \mathbf{k}) - 4(2\mathbf{i} + \mathbf{j}) = -4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k};$$

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -2\mathbf{i} - 3\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ -2 & -3 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k};$$

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = 2(2\mathbf{i} - \mathbf{j} + \mathbf{k}) - 3(\mathbf{i} + 2\mathbf{k}) = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$$

$$(d) \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ -1 & 0 & -1 \end{vmatrix} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 1 \\ 2 & 4 & -2 \end{vmatrix} = -10\mathbf{i} - 10\mathbf{k};$$

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = 10(-\mathbf{i} - \mathbf{k}) - 0(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = -10\mathbf{i} - 10\mathbf{k};$$

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix} = 4\mathbf{i} - 4\mathbf{j} - 4\mathbf{k} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 4 & -4 & -4 \end{vmatrix} = -12\mathbf{i} - 4\mathbf{j} - 8\mathbf{k};$$

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = 10(-\mathbf{i} - \mathbf{k}) - 1(2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = -12\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$$

13. (a) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} + (\mathbf{v} \cdot \mathbf{u})\mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} + (\mathbf{w} \cdot \mathbf{v})\mathbf{u} - (\mathbf{w} \cdot \mathbf{u})\mathbf{v} = \mathbf{0}$

(b) $[\mathbf{u} \cdot (\mathbf{v} \times \mathbf{i})]\mathbf{i} + [\mathbf{u} \cdot (\mathbf{v} \times \mathbf{j})]\mathbf{j} + [\mathbf{u} \cdot (\mathbf{v} \times \mathbf{k})]\mathbf{k} = [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{i}]\mathbf{i} + [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{j}]\mathbf{j} + [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{k}]\mathbf{k} = \mathbf{u} \times \mathbf{v}$

(c) $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{r}) = \mathbf{u} \cdot [\mathbf{v} \times (\mathbf{w} \times \mathbf{r})] = \mathbf{u} \cdot [(\mathbf{v} \cdot \mathbf{r})\mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{r}] = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{r}) - (\mathbf{u} \cdot \mathbf{r})(\mathbf{v} \cdot \mathbf{w})$

$$= \begin{vmatrix} \mathbf{u} \cdot \mathbf{w} & \mathbf{v} \cdot \mathbf{w} \\ \mathbf{u} \cdot \mathbf{r} & \mathbf{v} \cdot \mathbf{r} \end{vmatrix}$$

14. The formula is always true; $\mathbf{u} \times [\mathbf{u} \times (\mathbf{u} \times \mathbf{v})] \cdot \mathbf{w} = \mathbf{u} \times [(\mathbf{u} \cdot \mathbf{v})\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{v}] \cdot \mathbf{w}$

$$= [(\mathbf{u} \cdot \mathbf{v})\mathbf{u} \times \mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{u} \times \mathbf{v}] \cdot \mathbf{w} = -|\mathbf{u}|^2 \mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = -|\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$$

15. $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ is normal to the plane $x + 2y + 6z = 6$; $\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 2 & 6 \end{vmatrix} = 4\mathbf{i} - 5\mathbf{j} + \mathbf{k}$ is parallel to the

plane and perpendicular to the plane of $\mathbf{v}$ and $\mathbf{n} \Rightarrow \mathbf{w} = \mathbf{n} \times (\mathbf{v} \times \mathbf{n}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 6 \\ 4 & -5 & 1 \end{vmatrix} = 32\mathbf{i} + 23\mathbf{j} - 13\mathbf{k}$ is a

vector parallel to the plane $x + 2y + 6z = 6$ in the direction of the projection vector $\text{proj}_{\mathbf{P}} \mathbf{v}$. Therefore,

$$\text{proj}_{\mathbf{P}} \mathbf{v} = \text{proj}_{\mathbf{w}} \mathbf{v} = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|} \right) \frac{\mathbf{w}}{|\mathbf{w}|} = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2} \right) \mathbf{w} = \left(\frac{32 + 23 - 13}{32^2 + 23^2 + 13^2} \right) \mathbf{w} = \frac{42}{1722} \mathbf{w} = \frac{1}{41} \mathbf{w} = \frac{32}{41} \mathbf{i} + \frac{23}{41} \mathbf{j} - \frac{13}{41} \mathbf{k}$$

16. If $\mathbf{u} = (\cos A)\mathbf{i} + (\sin A)\mathbf{j}$ and $\mathbf{v} = (\cos B)\mathbf{i} + (\sin B)\mathbf{j}$, where $B > A$, then $\mathbf{u} \times \mathbf{v} = [|\mathbf{u}||\mathbf{v}| \sin(B - A)]\mathbf{k}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \end{vmatrix} = (\cos A \sin B - \sin A \cos B)\mathbf{k} \Rightarrow \sin(B - A) = \cos A \sin B - \sin A \cos B, \text{ since}$$

$$|\mathbf{u}| = 1 \text{ and } |\mathbf{v}| = 1.$$

17. (a) The vector from $(0, d)$ to $(kd, 0)$ is $\mathbf{r}_k = kd\mathbf{i} - d\mathbf{j} \Rightarrow |\mathbf{r}_k|^3 = \frac{1}{d^3(k^2 + 1)^{3/2}} \Rightarrow \frac{\mathbf{r}_k}{|\mathbf{r}_k|^3} = \frac{k\mathbf{i} - \mathbf{j}}{d^2(k^2 + 1)^{3/2}}$. The total force on the mass $(0, d)$ due to the masses Q_k for $k = -n, -n + 1, \dots, n - 1, n$ is

$$\mathbf{F} = \frac{GMm}{d^2}(-\mathbf{j}) + \frac{GMm}{2d^2}\left(\frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}\right) + \frac{GMm}{5d^2}\left(\frac{2\mathbf{i} - \mathbf{j}}{\sqrt{5}}\right) + \dots + \frac{GMm}{(n^2 + 1)d^2}\left(\frac{n\mathbf{i} - \mathbf{j}}{\sqrt{n^2 + 1}}\right) + \frac{GMm}{2d^2}\left(\frac{-\mathbf{i} - \mathbf{j}}{\sqrt{2}}\right) \\ + \frac{GMm}{5d^2}\left(\frac{-2\mathbf{i} - \mathbf{j}}{\sqrt{5}}\right) + \dots + \frac{GMm}{(n^2 + 1)d^2}\left(\frac{-n\mathbf{i} - \mathbf{j}}{\sqrt{n^2 + 1}}\right)$$

The $\mathbf{i}$ components cancel, giving

$$\mathbf{F} = \frac{GMm}{d^2} \left(-1 - \frac{2}{2\sqrt{2}} - \frac{2}{5\sqrt{5}} - \dots - \frac{2}{(n^2 + 1)(n^2 + 1)^{1/2}} \right) \mathbf{j} \Rightarrow \text{the magnitude of the force is}$$

$$|\mathbf{F}| = \frac{GMm}{d^2} \left(1 + \sum_{i=1}^n \frac{2}{(i^2+1)^{3/2}} \right).$$

(b) Yes, it is finite: $\lim_{n \rightarrow \infty} |\mathbf{F}| = \frac{GMm}{d^2} \left(1 + \sum_{i=1}^{\infty} \frac{2}{(i^2+1)^{3/2}} \right)$ is finite since $\sum_{i=1}^{\infty} \frac{2}{(i^2+1)^{3/2}}$ converges.

18. (a) If $\vec{x} \cdot \vec{y} = 0$, then $\vec{x} \times (\vec{x} \times \vec{y}) = (\vec{x} \cdot \vec{y})\vec{x} - (\vec{x} \cdot \vec{x})\vec{y} = -(\vec{x} \cdot \vec{x})\vec{y}$. This means that

$$\vec{x} \oplus \vec{y} = \vec{x} + \vec{y} + \frac{1}{c^2} \cdot \frac{1}{1 + \sqrt{1 - \frac{\vec{x} \cdot \vec{x}}{c^2}}} (-(\vec{x} \cdot \vec{x}))\vec{y} = \vec{x} + \left(1 - \frac{|\vec{x}|^2}{c^2 + \sqrt{c^4 - c^2|\vec{x}|^2}} \right) \vec{y}.$$

Since $\vec{x}$ and $\vec{y}$ are orthogonal, then $|\vec{x} \oplus \vec{y}|^2 = |\vec{x}|^2 + \left(1 - \frac{|\vec{x}|^2}{c^2 + \sqrt{c^4 - c^2|\vec{x}|^2}} \right)^2 |\vec{y}|^2$. A calculation will show that

$$|\vec{x}|^2 + \left(1 - \frac{|\vec{x}|^2}{c^2 + \sqrt{c^4 - c^2|\vec{x}|^2}} \right)^2 |\vec{y}|^2 < c^2.$$

$$\left(1 - \frac{|\vec{x}|^2}{c^2 + \sqrt{c^4 - c^2|\vec{x}|^2}} \right)^2 |\vec{y}|^2 < \left(1 - \frac{|\vec{x}|^2}{c^2 + \sqrt{c^4 - c^2|\vec{x}|^2}} \right) c^2.$$

$$|\vec{x} \oplus \vec{y}|^2 = |\vec{x}|^2 + \left(1 - \frac{|\vec{x}|^2}{c^2 + \sqrt{c^4 - c^2|\vec{x}|^2}} \right)^2 |\vec{y}|^2 < |\vec{x}|^2 + \left(1 - \frac{|\vec{x}|^2}{c^2 + \sqrt{c^4 - c^2|\vec{x}|^2}} \right) c^2 = c^2.$$

We now have $|\vec{x} \oplus \vec{y}|^2 < c^2$, so $|\vec{x} \oplus \vec{y}| < c$.

(b) If $\vec{x}$ and $\vec{y}$ are parallel, then $\vec{x} \times (\vec{x} \times \vec{y}) = \vec{0}$. This gives $\vec{x} \oplus \vec{y} = \frac{\vec{x} + \vec{y}}{1 + \frac{\vec{x} \cdot \vec{y}}{c^2}}$.

(i) If $\vec{x}$ and $\vec{y}$ have the same direction, then $\vec{x} \oplus \vec{y} = \frac{\vec{x} + \vec{y}}{1 + \frac{|\vec{x}||\vec{y}|}{c^2}}$ and $|\vec{x} \oplus \vec{y}| = \frac{|\vec{x}| + |\vec{y}|}{1 + \frac{|\vec{x}||\vec{y}|}{c^2}}$.

$$\text{Since } |\vec{y}| < c, |\vec{x}| < c, \text{ we have } |\vec{y}| \left(1 - \frac{|\vec{x}|}{c} \right) < c \left(1 - \frac{|\vec{x}|}{c} \right) \Rightarrow |\vec{y}| - \frac{|\vec{y}||\vec{x}|}{c} < c - |\vec{x}|$$

$$\Rightarrow |\vec{x}| + |\vec{y}| < c + \frac{|\vec{x}||\vec{y}|}{c} = c \left(1 + \frac{|\vec{x}||\vec{y}|}{c^2} \right) \Rightarrow \frac{|\vec{x}| + |\vec{y}|}{1 + \frac{|\vec{x}||\vec{y}|}{c^2}} < c. \text{ This means that } |\vec{x} \oplus \vec{y}| < c.$$

(ii) If $\vec{x}$ and $\vec{y}$ have opposite directions, then $\vec{x} \cdot \vec{y} = -|\vec{x}||\vec{y}|$ and $\vec{x} \oplus \vec{y} = \frac{\vec{x} + \vec{y}}{1 - \frac{|\vec{x}||\vec{y}|}{c^2}}$.

$$\text{Assume } |\vec{x}| \geq |\vec{y}|, \text{ then } |\vec{x} \oplus \vec{y}| = \frac{|\vec{x}| - |\vec{y}|}{1 - \frac{|\vec{x}||\vec{y}|}{c^2}}. \text{ Since } |\vec{x}| < c, \text{ we have } |\vec{x}| \left(1 + \frac{|\vec{y}|}{c} \right) < c \left(1 + \frac{|\vec{y}|}{c} \right)$$

$$\Rightarrow |\vec{x}| + \frac{|\vec{x}||\vec{y}|}{c} < c + |\vec{y}| \Rightarrow |\vec{x}| - |\vec{y}| < c - \frac{|\vec{x}||\vec{y}|}{c} = c \left(1 - \frac{|\vec{x}||\vec{y}|}{c^2} \right) \Rightarrow \frac{|\vec{x}| - |\vec{y}|}{1 - \frac{|\vec{x}||\vec{y}|}{c^2}} < c.$$

This means that $|\vec{x} \oplus \vec{y}| < c$. A similar argument holds if $|\vec{x}| > |\vec{y}|$.

(c) $\lim_{c \rightarrow \infty} \vec{x} \oplus \vec{y} = \vec{x} + \vec{y}$.

19. $r = \frac{(1+e)r_0}{1+e \cos \theta} \Rightarrow \frac{dr}{d\theta} = \frac{(1+e)r_0(e \sin \theta)}{(1+e \cos \theta)^2}$; $\frac{dr}{d\theta} = 0 \Rightarrow \frac{(1+e)r_0(e \sin \theta)}{(1+e \cos \theta)^2} = 0 \Rightarrow (1+e)r_0(e \sin \theta) = 0$

$\Rightarrow \sin \theta = 0 \Rightarrow \theta = 0$ or π . Note that $\frac{dr}{d\theta} > 0$ when $\sin \theta > 0$ and $\frac{dr}{d\theta} < 0$ when $\sin \theta < 0$. Since $\sin \theta < 0$ on

$-\pi < \theta < 0$ and $\sin \theta > 0$ on $0 < \theta < \pi$, r is a minimum when $\theta = 0$ and $r(0) = \frac{(1+e)r_0}{1+e \cos 0} = r_0$

20. (a) $f(x) = x - 1 - \frac{1}{2} \sin x = 0 \Rightarrow f(0) = -1$ and $f(2) = 2 - 1 - \frac{1}{2} \sin 2 \geq \frac{1}{2}$ since $|\sin 2| \leq 1$; since f is continuous on $[0, 2]$, the Intermediate Value Theorem implies there is a root between 0 and 2

(b) Root ≈ 1.4987011335179

21. (a) $\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$ and $\mathbf{v} = \frac{dr}{dt}\mathbf{u}_r + r \frac{d\theta}{dt}\mathbf{u}_\theta = \left(\frac{dr}{dt}\right)[(\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}] + \left(r \frac{d\theta}{dt}\right)[(-\sin \theta)\mathbf{i} + (\cos \theta)\mathbf{j}] \Rightarrow \mathbf{v} \cdot \mathbf{i} \frac{dx}{dt}$ and

$\mathbf{v} \cdot \mathbf{i} = \frac{dr}{dt} \cos \theta - r \frac{d\theta}{dt} \sin \theta \Rightarrow \frac{dx}{dt} = \frac{dr}{dt} \cos \theta - r \frac{d\theta}{dt} \sin \theta$; $\mathbf{v} \cdot \mathbf{j} = \frac{dy}{dt}$ and $\mathbf{v} \cdot \mathbf{j} = \frac{dr}{dt} \sin \theta + r \frac{d\theta}{dt} \cos \theta$

$\Rightarrow \frac{dy}{dt} = \frac{dr}{dt} \sin \theta + r \frac{d\theta}{dt} \cos \theta$

(b) $\mathbf{u}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} \Rightarrow \mathbf{v} \cdot \mathbf{u}_r = \frac{dx}{dt} \cos \theta + \frac{dy}{dt} \sin \theta$

$= \left(\frac{dr}{dt} \cos \theta - r \frac{d\theta}{dt} \sin \theta\right)(\cos \theta) + \left(\frac{dr}{dt} \sin \theta + r \frac{d\theta}{dt} \cos \theta\right)(\sin \theta)$ by part (a),

$\Rightarrow \mathbf{v} \cdot \mathbf{u}_r = \frac{dr}{dt}$; therefore, $\frac{dr}{dt} = \frac{dx}{dt} \cos \theta + \frac{dy}{dt} \sin \theta$;

$\mathbf{u}_\theta = (-\sin \theta)\mathbf{i} + (\cos \theta)\mathbf{j} \Rightarrow \mathbf{v} \cdot \mathbf{u}_\theta = -\frac{dx}{dt} \sin \theta + \frac{dy}{dt} \cos \theta$

$= \left(\frac{dr}{dt} \cos \theta - r \frac{d\theta}{dt} \sin \theta\right)(-\sin \theta) + \left(\frac{dr}{dt} \sin \theta + r \frac{d\theta}{dt} \cos \theta\right)(\cos \theta)$ by part (a) $\Rightarrow \mathbf{v} \cdot \mathbf{u}_\theta = r \frac{d\theta}{dt}$;

therefore, $r \frac{d\theta}{dt} = -\frac{dx}{dt} \sin \theta + \frac{dy}{dt} \cos \theta$

22. $\mathbf{r} = f(\theta) \Rightarrow \frac{dr}{dt} = f'(\theta) \frac{d\theta}{dt} \Rightarrow \frac{d^2r}{dt^2} = f''(\theta) \left(\frac{d\theta}{dt}\right)^2 + f'(\theta) \frac{d^2\theta}{dt^2}$; $\mathbf{v} = \frac{dr}{dt}\mathbf{u}_r + r \frac{d\theta}{dt}\mathbf{u}_\theta$

$= \left(\cos \theta \frac{dr}{dt} - r \sin \theta \frac{d\theta}{dt}\right)\mathbf{i} + \left(\sin \theta \frac{dr}{dt} + r \cos \theta \frac{d\theta}{dt}\right)\mathbf{j} \Rightarrow |\mathbf{v}| = \left[\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2\right]^{1/2} = \left[(f')^2 + f^2\right]^{1/2} \left(\frac{d\theta}{dt}\right)$;

$|\mathbf{v} \times \mathbf{a}| = |\dot{x}\ddot{y} - \dot{y}\ddot{x}|$, where $x = r \cos \theta$ and $y = r \sin \theta$. Then $\frac{dx}{dt} = (-r \sin \theta) \frac{d\theta}{dt} + (\cos \theta) \frac{dr}{dt}$

$\Rightarrow \frac{d^2x}{dt^2} = (-2 \sin \theta) \frac{d\theta}{dt} \frac{dr}{dt} - (r \cos \theta) \left(\frac{d\theta}{dt}\right)^2 - (r \sin \theta) \frac{d^2\theta}{dt^2} + (\cos \theta) \frac{d^2r}{dt^2}$; $\frac{dy}{dt} = (r \cos \theta) \frac{d\theta}{dt} + (\sin \theta) \frac{dr}{dt}$

$\Rightarrow \frac{d^2y}{dt^2} = (2 \cos \theta) \frac{d\theta}{dt} \frac{dr}{dt} - (r \sin \theta) \left(\frac{d\theta}{dt}\right)^2 + (r \cos \theta) \frac{d^2\theta}{dt^2} + (\sin \theta) \frac{d^2r}{dt^2}$. Then $|\mathbf{v} \times \mathbf{a}|$

$= (\text{after much algebra}) r^2 \left(\frac{d\theta}{dt}\right)^3 + r \frac{d^2\theta}{dt^2} \frac{dr}{dt} - r \frac{d\theta}{dt} \left(\frac{dr}{dt}\right)^2 \Rightarrow \kappa = \frac{r^2 \left(\frac{d\theta}{dt}\right)^3 + r \left(\frac{d^2\theta}{dt^2}\right) \left(\frac{dr}{dt}\right) - r \left(\frac{d\theta}{dt}\right) \left(\frac{dr}{dt}\right)^2}{\left[(f')^2 + f^2\right]^{3/2}}$

$$= \frac{r^2 + r \left(\frac{d^2\theta}{dt^2} \right) \left(\frac{dr}{dt} \right)^2 - r r'' - r' \left(\frac{d^2\theta}{dt^2} \right) \left(\frac{dr}{dt} \right)^2 + 2(r')^2}{[(r')^2 + r^2]^{3/2}} = \frac{r^2 - r r'' + 2(r')^2}{[(r')^2 + r^2]^{3/2}}$$

23. (a) Let $r = 2 - t$ and $\theta = 3t \Rightarrow \frac{dr}{dt} = -1$ and $\frac{d\theta}{dt} = 3 \Rightarrow \frac{d^2r}{dt^2} = \frac{d^2\theta}{dt^2} = 0$. The halfway point is $(1, 3) \Rightarrow t = 1$;

$$\mathbf{v} = \frac{dr}{dt} \mathbf{u}_r + r \frac{d\theta}{dt} \mathbf{u}_\theta \Rightarrow \mathbf{v}(1) = -\mathbf{u}_r + 3\mathbf{u}_\theta; \mathbf{a} = \left[\frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right] \mathbf{u}_r + \left[r \frac{d^2\theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right] \mathbf{u}_\theta \Rightarrow \mathbf{a}(1) = -9\mathbf{u}_r - 6\mathbf{u}_\theta$$

(b) It takes the beetle 2 min to crawl to the origin $\Rightarrow$ the rod has revolved 6 radians

$$\begin{aligned} \Rightarrow L &= \int_0^6 \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta = \int_0^6 \sqrt{\left(2 - \frac{\theta}{3}\right)^2 + \left(-\frac{1}{3}\right)^2} d\theta = \int_0^6 \sqrt{4 - \frac{4\theta}{3} + \frac{\theta^2}{9} + \frac{1}{9}} d\theta \\ &= \int_0^6 \sqrt{\frac{37 - 12\theta + \theta^2}{9}} d\theta = \frac{1}{3} \int_0^6 \sqrt{(\theta - 6)^2 + 1} d\theta = \frac{1}{3} \left[\frac{(\theta - 6)}{2} \sqrt{(\theta - 6)^2 + 1} + \frac{1}{2} \ln \left| \theta - 6 + \sqrt{(\theta - 6)^2 + 1} \right| \right]_0^6 \\ &= \sqrt{37} - \frac{1}{6} \ln(\sqrt{37} - 6) \approx 6.5 \text{ in.} \end{aligned}$$

24. $\mathbf{L}(t) = \mathbf{r}(t) + m\mathbf{v}(t) \Rightarrow \frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{r}}{dt} \times m\mathbf{v} \right) + \left(\mathbf{r} + m \frac{d^2\mathbf{r}}{dt^2} \right) \Rightarrow \frac{d\mathbf{L}}{dt} = (\mathbf{v} \times m\mathbf{v}) + (\mathbf{r} \times m\mathbf{a}) = \mathbf{r} \times m\mathbf{a}; \mathbf{F} = m\mathbf{a} \Rightarrow -\frac{c}{|\mathbf{r}|^3} \mathbf{r}$

$$= m\mathbf{a} \Rightarrow \frac{d\mathbf{L}}{dt} = \mathbf{r} \times m\mathbf{a} = \mathbf{r} \times \left(-\frac{c}{|\mathbf{r}|^3} \mathbf{r} \right) = -\frac{c}{|\mathbf{r}|^3} (\mathbf{r} \times \mathbf{r}) = \mathbf{0} \Rightarrow \mathbf{L} = \text{constant vector}$$