

CHAPTER 10 VECTORS AND MOTION IN SPACE

10.1 CARTESIAN (RECTANGULAR) COORDINATES AND VECTORS IN SPACE

1. The line through the point $(2, 3, 0)$ parallel to the z -axis
2. The line through the point $(-1, 0, 0)$ parallel to the y -axis
3. The x -axis
4. The line through the point $(1, 0, 0)$ parallel to the z -axis
5. The circle $x^2 + y^2 = 4$ in the plane $z = -2$
6. The circle $x^2 + z^2 = 4$ in the xz -plane
7. The circle $y^2 + z^2 = 1$ in the yz -plane
8. The circle $x^2 + z^2 = 9$ in the plane $y = -4$
9. The circle $x^2 + y^2 = 16$ in the xy -plane
10. The circle $x^2 + z^2 = 3$ in the xz -plane
11. (a) The first quadrant of the xy -plane
(b) The fourth quadrant of the xy -plane
12. (a) The slab bounded by the planes $x = 0$ and $x = 1$
(b) The square column bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$
(c) The unit cube in the first octant having one vertex at the origin
13. (a) The ball of radius 1 centered at the origin
(b) All points at distance greater than 1 unit from the origin
14. (a) The circumference and interior of the circle $x^2 + y^2 = 1$ in the xy -plane
(b) The circumference and interior of the circle $x^2 + y^2 = 1$ in the plane $z = 3$
(c) A solid cylindrical column of radius 1 whose axis is the z -axis
15. (a) The upper hemisphere of radius 1 centered at the origin
(b) The solid upper hemisphere of radius 1 centered at the origin
16. (a) The line $y = x$ in the xy -plane
(b) The plane $y = x$ consisting of all points of the form (x, x, z)
17. (a) $x = 3$
(b) $y = -1$
(c) $z = -2$
18. (a) $x = 3$
(b) $y = -1$
(c) $z = 2$
19. (a) $z = 1$
(b) $x = 3$
(c) $y = -1$
20. (a) $x^2 + y^2 = 4$, $z = 0$
(b) $y^2 + z^2 = 4$, $x = 0$
(c) $x^2 + z^2 = 4$, $y = 0$
21. (a) $x^2 + (y - 2)^2 = 4$, $z = 0$
(b) $(y - 2)^2 + z^2 = 4$, $x = 0$
(c) $x^2 + z^2 = 4$, $y = 2$
22. (a) $(x + 3)^2 + (y - 4)^2 = 1$, $z = 1$
(b) $(y - 4)^2 + (z - 1)^2 = 1$, $x = -3$
(c) $(x + 3)^2 + (z - 1)^2 = 1$, $y = 4$
23. (a) $y = 3$, $z = -1$
(b) $x = 1$, $z = -1$
(c) $x = 1$, $y = 3$
24. $\sqrt{x^2 + y^2 + z^2} = \sqrt{x^2 + (y - 2)^2 + z^2} \Rightarrow x^2 + y^2 + z^2 = x^2 + (y - 2)^2 + z^2 \Rightarrow y^2 = y^2 - 4y + 4 \Rightarrow y = 1$
25. $x^2 + y^2 + z^2 = 25$, $z = 3$
26. $x^2 + y^2 + (z - 1)^2 = 4$ and $x^2 + y^2 + (z + 1)^2 = 4 \Rightarrow x^2 + y^2 + (z - 1)^2 = x^2 + y^2 + (z + 1)^2 \Rightarrow z = 0$, $x^2 + y^2 = 3$
27. $0 \leq z \leq 1$
28. $0 \leq x \leq 2$, $0 \leq y \leq 2$, $0 \leq z \leq 2$

29. $z \leq 0$

30. $z = \sqrt{1-x^2-y^2}$

31. (a) $(x-1)^2 + (y-1)^2 + (z-1)^2 < 1$

(b) $(x-1)^2 + (y-1)^2 + (z-1)^2 > 1$

32. $1 \leq x^2 + y^2 + z^2 \leq 4$

33. length $= |2\mathbf{i} + \mathbf{j} - 2\mathbf{k}| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$, the direction is $\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k} \Rightarrow 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} = 3\left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right)$

34. length $= |9\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}| = \sqrt{81 + 4 + 36} = 11$, the direction is $\frac{9}{11}\mathbf{i} - \frac{2}{11}\mathbf{j} + \frac{6}{11}\mathbf{k} \Rightarrow 9\mathbf{i} - 2\mathbf{j} + 6\mathbf{k} = 11\left(\frac{9}{11}\mathbf{i} - \frac{2}{11}\mathbf{j} + \frac{6}{11}\mathbf{k}\right)$

35. length $= |5\mathbf{k}| = \sqrt{25} = 5$, the direction is $\mathbf{k} \Rightarrow 5\mathbf{k} = 5(\mathbf{k})$

36. length $= \left|\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}\right| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$, the direction is $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k} \Rightarrow \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k} = 1\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}\right)$

37. length $= \left|\frac{1}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}\right| = \sqrt{3\left(\frac{1}{\sqrt{6}}\right)^2} = \sqrt{\frac{1}{2}}$, the direction is $\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k} \Rightarrow \frac{1}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k} = \sqrt{\frac{1}{2}}\left(\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right)$

38. length $= \left|\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right| = \sqrt{3\left(\frac{1}{\sqrt{3}}\right)^2} = 1$, the direction is $\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} \Rightarrow \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} = 1\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right)$

39. (a) $2\mathbf{i}$ (b) $-\sqrt{3}\mathbf{k}$ (c) $\frac{3}{10}\mathbf{j} + \frac{2}{5}\mathbf{k}$ (d) $6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

40. (a) $-7\mathbf{j}$ (b) $-\frac{3\sqrt{2}}{5}\mathbf{i} - \frac{4\sqrt{2}}{5}\mathbf{k}$ (c) $\frac{1}{4}\mathbf{i} - \frac{1}{3}\mathbf{j} - \mathbf{k}$ (d) $\frac{a}{\sqrt{2}}\mathbf{i} + \frac{a}{\sqrt{3}}\mathbf{j} - \frac{a}{\sqrt{6}}\mathbf{k}$

41. $|\mathbf{v}| = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$; $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{13}\mathbf{v} = \frac{1}{13}(12\mathbf{i} - 5\mathbf{k}) \Rightarrow$ the desired vector is $\frac{7}{13}(12\mathbf{i} - 5\mathbf{k})$

42. $|\mathbf{v}| = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{3}}{2}$; $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k} \Rightarrow$ the desired vector is $-3\left(\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right) = -\sqrt{3}\mathbf{i} + \sqrt{3}\mathbf{j} + \sqrt{3}\mathbf{k}$

43. (a) the distance = the length $= |\vec{P_1P_2}| = |3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}| = \sqrt{9 + 16 + 25} = 5\sqrt{2}$

(b) $3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} = 5\sqrt{2}\left(\frac{3}{5\sqrt{2}}\mathbf{i} + \frac{4}{5\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{3}{5\sqrt{2}}\mathbf{i} + \frac{4}{5\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}$

(c) the midpoint is $\left(\frac{1}{2}, 3, \frac{5}{2}\right)$

44. (a) the distance = the length = $|\vec{P_1P_2}| = |3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}| = \sqrt{9 + 36 + 4} = 7$
 (b) $3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} = 7\left(\frac{3}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{3}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}$
 (c) the midpoint is $\left(\frac{5}{2}, 1, 6\right)$
45. (a) the distance = the length = $|\vec{P_1P_2}| = |-1\mathbf{i} - \mathbf{j} - \mathbf{k}| = \sqrt{3}$
 (b) $-1\mathbf{i} - \mathbf{j} - \mathbf{k} = \sqrt{3}\left(-\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right) \Rightarrow$ the direction is $-\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$
 (c) the midpoint is $\left(\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\right)$
46. (a) the distance = the length = $|\vec{P_1P_2}| = |2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}| = \sqrt{3 \cdot 2^2} = 2\sqrt{3}$
 (b) $2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} = 2\sqrt{3}\left(\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$
 (c) the midpoint is $(1, -1, -1)$
47. $\vec{AB} = (5 - a)\mathbf{i} + (1 - b)\mathbf{j} + (3 - c)\mathbf{k} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \Rightarrow 5 - a = 1, 1 - b = 4, \text{ and } 3 - c = -2 \Rightarrow a = 4, b = -3, \text{ and } c = 5 \Rightarrow A$ is the point $(4, -3, 5)$
48. $\vec{AB} = (a + 2)\mathbf{i} + (b + 3)\mathbf{j} + (c - 6)\mathbf{k} = -7\mathbf{i} + 3\mathbf{j} + 8\mathbf{k} \Rightarrow a + 2 = -7, b + 3 = 3, \text{ and } c - 6 = 8 \Rightarrow a = -9, b = 0, \text{ and } c = 14 \Rightarrow B$ is the point $(-9, 0, 14)$
49. $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 14$ 50. $x^2 + (y + 1)^2 + (z - 5)^2 = 4$
51. center $(-2, 0, 2)$, radius $2\sqrt{2}$ 52. center $\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$, radius $\frac{\sqrt{21}}{2}$
53. $x^2 + y^2 + z^2 + 4x - 4z = 0 \Rightarrow (x^2 + 4x + 4) + y^2 + (z^2 - 4z + 4) = 4 + 4 \Rightarrow (x + 2)^2 + (y - 0)^2 + (z - 2)^2 = (\sqrt{8})^2$
 $\Rightarrow$ the center is at $(-2, 0, 2)$ and the radius is $2\sqrt{2}$
54. $x^2 + y^2 + z^2 - 6y + 8z = 0 \Rightarrow x^2 + (y^2 - 6y + 9) + (z^2 + 8z + 16) = 9 + 16 \Rightarrow (x - 0)^2 + (y - 3)^2 + (z + 4)^2 = 5^2$
 $\Rightarrow$ the center is at $(0, 3, -4)$ and the radius is 5
55. $2x^2 + 2y^2 + 2z^2 + x + y + z = 9 \Rightarrow x^2 + \frac{1}{2}x + y^2 + \frac{1}{2}y + z^2 + \frac{1}{2}z = \frac{9}{2}$
 $\Rightarrow \left(x^2 + \frac{1}{2}x + \frac{1}{16}\right) + \left(y^2 + \frac{1}{2}y + \frac{1}{16}\right) + \left(z^2 + \frac{1}{2}z + \frac{1}{16}\right) = \frac{9}{2} + \frac{3}{16} = \frac{75}{16} \Rightarrow \left(x + \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 + \left(z + \frac{1}{4}\right)^2 = \left(\frac{5\sqrt{3}}{4}\right)^2$
 $\Rightarrow$ the center is at $\left(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$ and the radius is $\frac{5\sqrt{3}}{4}$
56. $3x^2 + 3y^2 + 3z^2 + 2y - 2z = 9 \Rightarrow x^2 + y^2 + \frac{2}{3}y + z^2 - \frac{2}{3}z = 3 \Rightarrow x^2 + \left(y^2 + \frac{2}{3}y + \frac{1}{9}\right) + \left(z^2 - \frac{2}{3}z + \frac{1}{9}\right) = 3 + \frac{2}{9}$
 $\Rightarrow (x - 0)^2 + \left(y + \frac{1}{3}\right)^2 + \left(z - \frac{1}{3}\right)^2 = \left(\frac{\sqrt{29}}{3}\right)^2 \Rightarrow$ the center is at $\left(0, -\frac{1}{3}, \frac{1}{3}\right)$ and the radius is $\frac{\sqrt{29}}{3}$

57. (a) the distance between (x, y, z) and $(x, 0, 0)$ is $\sqrt{y^2 + z^2}$
 (b) the distance between (x, y, z) and $(0, y, 0)$ is $\sqrt{x^2 + z^2}$
 (c) the distance between (x, y, z) and $(0, 0, z)$ is $\sqrt{x^2 + y^2}$
58. (a) the distance between (x, y, z) and $(x, y, 0)$ is $|z|$
 (b) the distance between (x, y, z) and $(0, y, z)$ is $|x|$
 (c) the distance between (x, y, z) and $(x, 0, z)$ is $|y|$
59. (a) the midpoint of AB is $M\left(\frac{5}{2}, \frac{5}{2}, 0\right)$ and $\vec{CM} = \left(\frac{5}{2} - 1\right)\mathbf{i} + \left(\frac{5}{2} - 1\right)\mathbf{j} + (0 - 3)\mathbf{k} = \frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}$
 (b) the desired vector is $\left(\frac{2}{3}\right)\vec{CM} = \frac{2}{3}\left(\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}\right) = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$
 (c) the vector whose sum is the vector from the origin to C and the result of part (b) will terminate at the center of mass $\Rightarrow$ the terminal point of $(\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ is the point $(2, 2, 1)$, which is the location of the center of mass
60. The midpoint of AB is $M\left(\frac{3}{2}, 0, \frac{5}{2}\right)$ and $\left(\frac{2}{3}\right)\vec{CM} = \frac{2}{3}\left[\left(\frac{3}{2} + 1\right)\mathbf{i} + (0 - 2)\mathbf{j} + \left(\frac{5}{2} + 1\right)\mathbf{k}\right] = \frac{2}{3}\left(\frac{5}{2}\mathbf{i} - 2\mathbf{j} + \frac{7}{2}\mathbf{k}\right)$
 $= \frac{5}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}$. The terminal point of $\left(\frac{5}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}\right) + \vec{OC} = \left(\frac{5}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}\right) + (-\mathbf{i} + 2\mathbf{j} - \mathbf{k})$
 $= \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{4}{3}\mathbf{k}$ is the point $\left(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}\right)$ which is the location of the intersection of the medians.
61. Without loss of generality we identify the vertices of the quadrilateral such that $A(0, 0, 0)$, $B(x_b, 0, 0)$, $C(x_c, y_c, 0)$ and $D(x_d, y_d, z_d) \Rightarrow$ the midpoint of AB is $M_{AB}\left(\frac{x_b}{2}, 0, 0\right)$, the midpoint of BC is $M_{BC}\left(\frac{x_b + x_c}{2}, \frac{y_c}{2}, 0\right)$, the midpoint of CD is $M_{CD}\left(\frac{x_c + x_d}{2}, \frac{y_c + y_d}{2}, \frac{z_d}{2}\right)$ and the midpoint of AD is $M_{AD}\left(\frac{x_d}{2}, \frac{y_d}{2}, \frac{z_d}{2}\right) \Rightarrow$ the midpoint of $M_{AB}M_{CD}$ is $\left(\frac{\frac{x_b}{2} + \frac{x_c + x_d}{2}}{2}, \frac{\frac{y_c}{2} + \frac{y_c + y_d}{2}}{2}, \frac{\frac{z_d}{2}}{2}\right)$ which is the same as the midpoint of $M_{AD}M_{BC} = \left(\frac{\frac{x_b + x_c}{2} + \frac{x_d}{2}}{2}, \frac{\frac{y_c + y_d}{2}}{2}, \frac{\frac{z_d}{4}}{2}\right)$.
62. Let $V_1, V_2, V_3, \dots, V_n$ be the vertices of a regular n -sided polygon and $\mathbf{v}_i$ denote the vector from the center to V_i for $i = 1, 2, 3, \dots, n$. If $\mathbf{S} = \sum_{i=1}^n \mathbf{v}_i$ and the polygon is rotated through an angle of $\frac{i(2\pi)}{n}$ where $i = 1, 2, 3, \dots, n$, then $\mathbf{S}$ would remain the same. Since $\mathbf{S}$ does not change with these rotations we conclude that $\mathbf{S} = \mathbf{0}$.
63. Without loss of generality we can coordinatize the vertices of the triangle such that $A(0, 0)$, $B(b, 0)$ and $C(x_c, y_c) \Rightarrow a$ is located at $\left(\frac{b + x_c}{2}, \frac{y_c}{2}\right)$, b is at $\left(\frac{x_c}{2}, \frac{y_c}{2}\right)$ and c is at $\left(\frac{b}{2}, 0\right)$. Therefore, $\vec{Aa} = \left(\frac{b}{2} + \frac{x_c}{2}\right)\mathbf{i} + \left(\frac{y_c}{2}\right)\mathbf{j}$, $\vec{Bb} = \left(\frac{x_c}{2} - b\right)\mathbf{i} + \left(\frac{y_c}{2}\right)\mathbf{j}$, and $\vec{Cc} = \left(\frac{b}{2} - x_c\right)\mathbf{i} + (-y_c)\mathbf{j} \Rightarrow \vec{Aa} + \vec{Bb} + \vec{Cc} = \mathbf{0}$.

10.2 DOT AND CROSS PRODUCTS

NOTE: In Exercises 1-6 below we calculate $\text{proj}_{\mathbf{v}} \mathbf{u}$ as the vector $\left(\frac{|\mathbf{u}| \cos \theta}{|\mathbf{v}|}\right) \mathbf{v}$, so the scalar multiplier of $\mathbf{v}$ is

the number in column 5 divided by the number in column 2.

	$\mathbf{v} \cdot \mathbf{u}$	$ \mathbf{v} $	$ \mathbf{u} $	$\cos \theta$	$ \mathbf{u} \cos \theta$	$\text{proj}_{\mathbf{v}} \mathbf{u}$
1.	-25	5	5	-1	-5	$-2\mathbf{i} + 4\mathbf{j} - \sqrt{5}\mathbf{k}$
2.	3	1	13	$\frac{3}{13}$	3	$3\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}\right)$
3.	25	15	5	$\frac{1}{3}$	$\frac{5}{3}$	$\frac{1}{9}(10\mathbf{i} + 11\mathbf{j} - 2\mathbf{k})$
4.	13	15	3	$\frac{13}{45}$	$\frac{13}{15}$	$\frac{13}{225}(2\mathbf{i} + 10\mathbf{j} - 11\mathbf{k})$
5.	2	$\sqrt{34}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}\sqrt{34}}$	$\frac{2}{\sqrt{34}}$	$\frac{1}{17}(5\mathbf{j} - 3\mathbf{k})$
6.	$\sqrt{3} - \sqrt{2}$	$\sqrt{2}$	3	$\frac{\sqrt{3} - \sqrt{2}}{3\sqrt{2}}$	$\frac{\sqrt{3} - \sqrt{2}}{\sqrt{2}}$	$\frac{\sqrt{3} - \sqrt{2}}{2}(-\mathbf{i} + \mathbf{j})$
7.	$\mathbf{u} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} + \left(\mathbf{u} - \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}\right) = \frac{3}{2}(\mathbf{i} + \mathbf{j}) + \left[(3\mathbf{j} + 4\mathbf{k}) - \frac{3}{2}(\mathbf{i} + \mathbf{j})\right] = \left(\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}\right) + \left(-\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} + 4\mathbf{k}\right)$, where $\mathbf{v} \cdot \mathbf{u} = 3$ and $\mathbf{v} \cdot \mathbf{v} = 2$					
8.	$\mathbf{u} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} + \left(\mathbf{u} - \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}\right) = \frac{1}{2}\mathbf{v} + \left(\mathbf{u} - \frac{1}{2}\mathbf{v}\right) = \frac{1}{2}(\mathbf{i} + \mathbf{j}) + \left[(\mathbf{j} + \mathbf{k}) - \frac{1}{2}(\mathbf{i} + \mathbf{j})\right] = \left(\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right) + \left(-\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \mathbf{k}\right)$, where $\mathbf{v} \cdot \mathbf{u} = 1$ and $\mathbf{v} \cdot \mathbf{v} = 2$					
9.	$\mathbf{u} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} + \left(\mathbf{u} - \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}\right) = \frac{14}{3}(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + \left[(8\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}) - \left(\frac{14}{3}\mathbf{i} + \frac{28}{3}\mathbf{j} - \frac{14}{3}\mathbf{k}\right)\right] = \left(\frac{14}{3}\mathbf{i} + \frac{28}{3}\mathbf{j} - \frac{14}{3}\mathbf{k}\right) + \left(\frac{10}{3}\mathbf{i} - \frac{16}{3}\mathbf{j} - \frac{22}{3}\mathbf{k}\right)$, where $\mathbf{v} \cdot \mathbf{u} = 28$ and $\mathbf{v} \cdot \mathbf{v} = 6$					
10.	$\mathbf{u} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} + \left(\mathbf{u} - \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}\right) = \frac{1}{1}(\mathbf{v}) + \left[(\mathbf{i} + \mathbf{j} + \mathbf{k}) - \left(\frac{1}{1}\right)\mathbf{v}\right] = (\mathbf{i}) + (\mathbf{j} + \mathbf{k})$, where $\mathbf{v} \cdot \mathbf{u} = 1$ and $\mathbf{v} \cdot \mathbf{v} = 1$; yes					
11.	$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{ \mathbf{u} \mathbf{v} }\right) = \cos^{-1}\left(\frac{(2)(1) + (1)(2) + (0)(-1)}{\sqrt{2^2 + 1^2 + 0^2} \sqrt{1^2 + 2^2 + (-1)^2}}\right) = \cos^{-1}\left(\frac{4}{\sqrt{5} \sqrt{6}}\right) = \cos^{-1}\left(\frac{4}{\sqrt{30}}\right) \approx 0.75 \text{ rad}$					
12.	$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{ \mathbf{u} \mathbf{v} }\right) = \cos^{-1}\left(\frac{(2)(3) + (-2)(0) + (1)(4)}{\sqrt{2^2 + (-2)^2 + 1^2} \sqrt{3^2 + 0^2 + 4^2}}\right) = \cos^{-1}\left(\frac{10}{\sqrt{9} \sqrt{25}}\right) = \cos^{-1}\left(\frac{2}{3}\right) \approx 0.84 \text{ rad}$					
13.	$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{ \mathbf{u} \mathbf{v} }\right) = \cos^{-1}\left(\frac{(\sqrt{3})(\sqrt{3}) + (-7)(1) + (0)(-2)}{\sqrt{(\sqrt{3})^2 + (-7)^2 + 0^2} \sqrt{(\sqrt{3})^2 + (1)^2 + (-2)^2}}\right) = \cos^{-1}\left(\frac{3-7}{\sqrt{52} \sqrt{8}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{26}}\right) \approx 1.77 \text{ rad}$					

$$14. \theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{(1)(-1) + (\sqrt{2})(1) + (-\sqrt{2})(1)}{\sqrt{(1)^2 + (\sqrt{2})^2 + (-\sqrt{2})^2} \sqrt{(-1)^2 + (1)^2 + (1)^2}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{5}\sqrt{3}}\right) \\ = \cos^{-1}\left(\frac{-1}{\sqrt{15}}\right) \approx 1.83 \text{ rad}$$

$$15. (a) \cos \alpha = \frac{\mathbf{i} \cdot \mathbf{v}}{|\mathbf{i}||\mathbf{v}|} = \frac{a}{|\mathbf{v}|}, \cos \beta = \frac{\mathbf{j} \cdot \mathbf{v}}{|\mathbf{j}||\mathbf{v}|} = \frac{b}{|\mathbf{v}|}, \cos \gamma = \frac{\mathbf{k} \cdot \mathbf{v}}{|\mathbf{k}||\mathbf{v}|} = \frac{c}{|\mathbf{v}|} \text{ and}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \left(\frac{a}{|\mathbf{v}|}\right)^2 + \left(\frac{b}{|\mathbf{v}|}\right)^2 + \left(\frac{c}{|\mathbf{v}|}\right)^2 = \frac{a^2 + b^2 + c^2}{|\mathbf{v}|^2} = \frac{|\mathbf{v}||\mathbf{v}|}{|\mathbf{v}|^2} = 1$$

$$(b) |\mathbf{v}| = 1 \Rightarrow \cos \alpha = \frac{a}{|\mathbf{v}|} = a, \cos \beta = \frac{b}{|\mathbf{v}|} = b \text{ and } \cos \gamma = \frac{c}{|\mathbf{v}|} = c \text{ are the direction cosines of } \mathbf{v}$$

$$16. \mathbf{u} = 10\mathbf{i} + 2\mathbf{k} \text{ is parallel to the pipe in the north direction and } \mathbf{v} = 10\mathbf{j} + \mathbf{k} \text{ is parallel to the pipe in the east direction. The angle between the two pipes is } \theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{2}{\sqrt{104}\sqrt{101}}\right) \approx 1.55 \text{ rad} \approx 88.88^\circ.$$

$$17. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -1 \\ 1 & 0 & -1 \end{vmatrix} = 3\left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right) \Rightarrow \text{length} = 3 \text{ and the direction is } \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k};$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -3\left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right) \Rightarrow \text{length} = 3 \text{ and the direction is } -\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

$$18. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ -1 & 1 & 0 \end{vmatrix} = 5(\mathbf{k}) \Rightarrow \text{length} = 5 \text{ and the direction is } \mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -5(\mathbf{k}) \Rightarrow \text{length} = 5 \text{ and the direction is } -\mathbf{k}$$

$$19. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix} = \mathbf{0} \Rightarrow \text{length} = 0 \text{ and has no direction}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = \mathbf{0} \Rightarrow \text{length} = 0 \text{ and has no direction}$$

$$20. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} = \mathbf{0} \Rightarrow \text{length} = 0 \text{ and has no direction}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = \mathbf{0} \Rightarrow \text{length} = 0 \text{ and has no direction}$$

$$21. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 0 & -3 & 0 \end{vmatrix} = -6(\mathbf{k}) \Rightarrow \text{length} = 6 \text{ and the direction is } -\mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 6(\mathbf{k}) \Rightarrow \text{length} = 6 \text{ and the direction is } \mathbf{k}$$

$$22. \mathbf{u} \times \mathbf{v} = (\mathbf{i} \times \mathbf{j}) \times (\mathbf{j} \times \mathbf{k}) = \mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j} \Rightarrow \text{length} = 1 \text{ and the direction is } \mathbf{j}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -\mathbf{j} \Rightarrow \text{length} = 1 \text{ and the direction is } -\mathbf{j}$$

$$23. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & -2 & -4 \\ 2 & 2 & 1 \end{vmatrix} = 6\mathbf{i} - 12\mathbf{k} \Rightarrow \text{length} = 6\sqrt{5} \text{ and the direction is } \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{k}$$

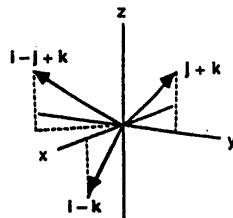
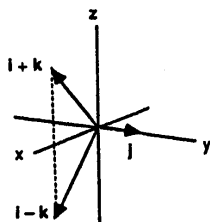
$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -(6\mathbf{i} - 12\mathbf{k}) \Rightarrow \text{length} = 6\sqrt{5} \text{ and the direction is } -\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}$$

$$24. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{2} & -\frac{1}{2} & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \Rightarrow \text{length} = 2\sqrt{3} \text{ and the direction is } -\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

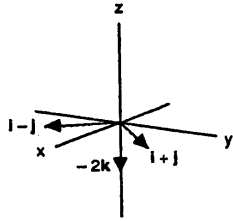
$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -(-2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \Rightarrow \text{length} = 2\sqrt{3} \text{ and the direction is } \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$$

$$25. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{i} + \mathbf{k}$$

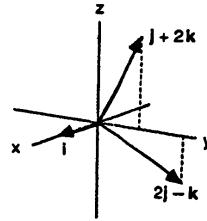
$$26. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k}$$



$$27. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = -2\mathbf{k}$$



$$28. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 2\mathbf{j} - \mathbf{k}$$



$$29. (a) \vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} = 8\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} \Rightarrow \text{Area} = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2} \sqrt{64 + 16 + 16} = 2\sqrt{6}$$

$$(b) \mathbf{u} = \pm \frac{\vec{PQ} \times \vec{PR}}{|\vec{PQ} \times \vec{PR}|} = \pm \frac{1}{\sqrt{6}} (2\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$30. (a) \vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{vmatrix} = 4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \Rightarrow \text{Area} = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2} \sqrt{16 + 16 + 4} = 3$$

$$(b) \mathbf{u} = \pm \frac{\vec{PQ} \times \vec{PR}}{|\vec{PQ} \times \vec{PR}|} = \pm \frac{1}{3} (2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$

$$31. (a) \vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{j} \Rightarrow \text{Area} = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2} \sqrt{1 + 1} = \frac{\sqrt{2}}{2}$$

$$(b) \mathbf{u} = \pm \frac{\vec{PQ} \times \vec{PR}}{|\vec{PQ} \times \vec{PR}|} = \pm \frac{1}{\sqrt{2}} (-\mathbf{i} + \mathbf{j}) = \pm \frac{1}{\sqrt{2}} (\mathbf{i} - \mathbf{j})$$

$$32. (a) \vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 1 & 0 & -2 \end{vmatrix} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow \text{Area} = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2} \sqrt{4 + 9 + 1} = \frac{\sqrt{14}}{2}$$

$$(b) \mathbf{u} = \pm \frac{\vec{PQ} \times \vec{PR}}{|\vec{PQ} \times \vec{PR}|} = \pm \frac{1}{\sqrt{14}} (2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$$

$$33. \text{ If } \mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}, \text{ and } \mathbf{w} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}, \text{ then } \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

$$\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \text{ and } \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \text{ which all have the same value, since the}$$

interchanging of two pair of rows in a determinant does not change its value $\Rightarrow$ the volume is

$$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs} \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$$

$$34. |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs} \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 4 \text{ (for details about verification, see Exercise 33)}$$

$$35. |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs} \begin{vmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = |-7| = 7 \text{ (for details about verification, see Exercise 33)}$$

$$36. |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs} \begin{vmatrix} 1 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix} = 8 \text{ (for details about verification, see Exercise 33)}$$

$$37. (a) \mathbf{u} \cdot \mathbf{v} = -6, \mathbf{u} \cdot \mathbf{w} = -81, \mathbf{v} \cdot \mathbf{w} = 18 \Rightarrow \text{none}$$

$$(b) \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ 0 & 1 & -5 \end{vmatrix} \neq \mathbf{0}, \mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ -15 & 3 & -3 \end{vmatrix} = \mathbf{0}, \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -5 \\ -15 & 3 & -3 \end{vmatrix} \neq \mathbf{0}$$

$\Rightarrow \mathbf{u}$ and $\mathbf{w}$ are parallel

$$38. (a) \mathbf{u} \cdot \mathbf{v} = 0, \mathbf{u} \cdot \mathbf{w} = 0, \mathbf{u} \cdot \mathbf{r} = -3\pi, \mathbf{v} \cdot \mathbf{w} = 0, \mathbf{v} \cdot \mathbf{r} = 0, \mathbf{w} \cdot \mathbf{r} = 0 \Rightarrow \mathbf{u} \perp \mathbf{v}, \mathbf{u} \perp \mathbf{w}, \mathbf{v} \perp \mathbf{w}, \mathbf{v} \perp \mathbf{r} \text{ and } \mathbf{w} \perp \mathbf{r}$$

$$(b) \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -1 & 1 & 1 \end{vmatrix} \neq \mathbf{0}, \mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{vmatrix} \neq \mathbf{0}, \mathbf{u} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -\frac{\pi}{2} & -\pi & \frac{\pi}{2} \end{vmatrix} = \mathbf{0}$$

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \neq \mathbf{0}, \mathbf{v} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ -\frac{\pi}{2} & -\pi & \frac{\pi}{2} \end{vmatrix} \neq \mathbf{0}, \mathbf{w} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ -\frac{\pi}{2} & -\pi & \frac{\pi}{2} \end{vmatrix} \neq \mathbf{0}$$

$\Rightarrow \mathbf{u}$ and $\mathbf{r}$ are parallel

$$39. |\vec{PQ} \times \mathbf{F}| = |\vec{PQ}| |\mathbf{F}| \sin(60^\circ) = \frac{2}{3} \cdot 30 \cdot \frac{\sqrt{3}}{2} \text{ ft} \cdot \text{lb} = 10\sqrt{3} \text{ ft} \cdot \text{lb}$$

$$40. |\vec{PQ} \times \mathbf{F}| = |\vec{PQ}| |\mathbf{F}| \sin(135^\circ) = \frac{2}{3} \cdot 30 \cdot \frac{\sqrt{2}}{2} \text{ ft} \cdot \text{lb} = 10\sqrt{2} \text{ ft} \cdot \text{lb}$$

$$41. (a) \text{ true, } |\mathbf{u}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

$$(b) \text{ not always true, } \mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$$

$$(c) \text{ true, } \mathbf{u} \times \mathbf{0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

$$(d) \text{ true, } \mathbf{u} \times (-\mathbf{u}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ -a_1 & -a_2 & -a_3 \end{vmatrix} = (-a_2a_3 + a_2a_3)\mathbf{i} + (-a_1a_3 + a_1a_3)\mathbf{j} + (-a_1a_2 + a_1a_2)\mathbf{k} = \mathbf{0}$$

$$(e) \text{ not always true, } \mathbf{i} \times \mathbf{j} = \mathbf{k} \neq -\mathbf{k} = \mathbf{j} \times \mathbf{i} \text{ for example}$$

$$(f) \text{ true, distributive property of the cross product}$$

$$(g) \text{ true, } (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{v}) = \mathbf{u} \cdot \mathbf{0} = 0$$

$$(h) \text{ true, the volume of a parallelepiped with } \mathbf{u}, \mathbf{v}, \text{ and } \mathbf{w} \text{ along the three edges is } (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}), \text{ since the dot product is commutative.}$$

$$42. (a) \text{ true, } \mathbf{u} \cdot \mathbf{v} = a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3 = \mathbf{v} \cdot \mathbf{u}$$

$$(b) \text{ true, } \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = -(\mathbf{v} \times \mathbf{u})$$

$$(c) \text{ true, } (-\mathbf{u}) \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a_1 & -a_2 & -a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = -(\mathbf{u} \times \mathbf{v})$$

$$(d) \text{ true, } (c\mathbf{u}) \cdot \mathbf{v} = (ca_1)b_1 + (ca_2)b_2 + (ca_3)b_3 = a_1(cb_1) + a_2(cb_2) + a_3(cb_3) = \mathbf{u} \cdot (c\mathbf{v}) = c(a_1b_1 + a_2b_2 + a_3b_3) = c(\mathbf{u} \cdot \mathbf{v})$$

$$(e) \text{ true, } c(\mathbf{u} \times \mathbf{v}) = c \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ ca_1 & ca_2 & ca_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (c\mathbf{u}) \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ cb_1 & cb_2 & cb_3 \end{vmatrix} = \mathbf{u} \times (c\mathbf{v})$$

- (f) true, $\mathbf{u} \cdot \mathbf{u} = a_1^2 + a_2^2 + a_3^2 = \left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)^2 = |\mathbf{u}|^2$
- (g) true, $(\mathbf{u} \times \mathbf{u}) \cdot \mathbf{u} = \mathbf{0} \cdot \mathbf{u} = 0$
- (h) true, $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}$ and $\mathbf{u} \times \mathbf{v} \perp \mathbf{v} \Rightarrow (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$
43. (a) $\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v}$ (b) $\pm(\mathbf{u} \times \mathbf{v})$ (c) $\pm(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ (d) $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$
44. (a) $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{u} \times \mathbf{w})$
- (b) $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v}) = (\mathbf{u} + \mathbf{v}) \times \mathbf{u} - (\mathbf{u} + \mathbf{v}) \times \mathbf{v} = \mathbf{u} \times \mathbf{u} + \mathbf{v} \times \mathbf{u} - \mathbf{u} \times \mathbf{v} - \mathbf{v} \times \mathbf{v}$
 $= \mathbf{0} + \mathbf{v} \times \mathbf{u} - \mathbf{u} \times \mathbf{v} - \mathbf{0} = 2(\mathbf{v} \times \mathbf{u})$, or simply $\mathbf{u} \times \mathbf{v}$
- (c) $|\mathbf{u}| \frac{|\mathbf{v}|}{|\mathbf{v}|}$ (d) $|\mathbf{u} \times \mathbf{w}|$
45. (a) yes, $\mathbf{u} \times \mathbf{v}$ and $\mathbf{w}$ are both vectors (b) no, $\mathbf{u}$ is a vector but $\mathbf{v} \cdot \mathbf{w}$ is a scalar
- (c) yes, $\mathbf{u}$ and $\mathbf{u} \times \mathbf{w}$ are both vectors (d) no, $\mathbf{u}$ is a vector but $\mathbf{v} \cdot \mathbf{w}$ is a scalar
46. $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ is perpendicular to $\mathbf{u} \times \mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ is parallel to a vector in the plane of $\mathbf{u}$ and $\mathbf{v}$ which means it lies in the plane determined by $\mathbf{u}$ and $\mathbf{v}$. The situation is degenerate if $\mathbf{u}$ and $\mathbf{v}$ are parallel so $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and the vectors do not determine a plane. Similar reasoning shows that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ lies in the plane of $\mathbf{v}$ and $\mathbf{w}$ provided $\mathbf{v}$ and $\mathbf{w}$ are nonparallel.
47. No, $\mathbf{v}$ need not equal $\mathbf{w}$. For example, $\mathbf{i} + \mathbf{j} \neq -\mathbf{i} + \mathbf{j}$, but $\mathbf{i} \times (\mathbf{i} + \mathbf{j}) = \mathbf{i} \times \mathbf{i} + \mathbf{i} \times \mathbf{j} = \mathbf{0} + \mathbf{k} = \mathbf{k}$ and $\mathbf{i} \times (-\mathbf{i} + \mathbf{j}) = -\mathbf{i} \times \mathbf{i} + \mathbf{i} \times \mathbf{j} = \mathbf{0} + \mathbf{k} = \mathbf{k}$.
48. Yes. If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then $\mathbf{u} \times (\mathbf{v} - \mathbf{w}) = \mathbf{0}$ and $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = 0$. Suppose now that $\mathbf{v} \neq \mathbf{w}$. Then $\mathbf{u} \times (\mathbf{v} - \mathbf{w}) = \mathbf{0}$ implies that $\mathbf{v} - \mathbf{w} = k\mathbf{u}$ for some real number $k \neq 0$. This in turn implies that $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot (k\mathbf{u}) = k|\mathbf{u}|^2 = 0$, which implies that $\mathbf{u} = \mathbf{0}$. Since $\mathbf{u} \neq \mathbf{0}$, it cannot be true that $\mathbf{v} \neq \mathbf{w}$, so $\mathbf{v} = \mathbf{w}$.

$$49. \vec{\mathbf{AB}} = -\mathbf{i} + \mathbf{j} \text{ and } \vec{\mathbf{AD}} = -\mathbf{i} - \mathbf{j} \Rightarrow \vec{\mathbf{AB}} \times \vec{\mathbf{AD}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & -1 & 0 \end{vmatrix} = 2\mathbf{k} \Rightarrow \text{area} = |\vec{\mathbf{AB}} \times \vec{\mathbf{AD}}| = 2$$

$$50. \vec{\mathbf{AB}} = 7\mathbf{i} + 3\mathbf{j} \text{ and } \vec{\mathbf{AD}} = 2\mathbf{i} + 5\mathbf{j} \Rightarrow \vec{\mathbf{AB}} \times \vec{\mathbf{AD}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & 0 \\ 2 & 5 & 0 \end{vmatrix} = 29\mathbf{k} \Rightarrow \text{area} = |\vec{\mathbf{AB}} \times \vec{\mathbf{AD}}| = 29$$

$$51. \vec{\mathbf{AB}} = 3\mathbf{i} - 2\mathbf{j} \text{ and } \vec{\mathbf{AD}} = 5\mathbf{i} + \mathbf{j} \Rightarrow \vec{\mathbf{AB}} \times \vec{\mathbf{AD}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 0 \\ 5 & 1 & 0 \end{vmatrix} = 13\mathbf{k} \Rightarrow \text{area} = |\vec{\mathbf{AB}} \times \vec{\mathbf{AD}}| = 13$$

$$52. \vec{\mathbf{AB}} = 7\mathbf{i} - 4\mathbf{j} \text{ and } \vec{\mathbf{AD}} = 2\mathbf{i} + 5\mathbf{j} \Rightarrow \vec{\mathbf{AB}} \times \vec{\mathbf{AD}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -4 & 0 \\ 2 & 5 & 0 \end{vmatrix} = 43\mathbf{k} \Rightarrow \text{area} = |\vec{\mathbf{AB}} \times \vec{\mathbf{AD}}| = 43$$

$$53. \vec{AB} = -2\mathbf{i} + 3\mathbf{j} \text{ and } \vec{AC} = 3\mathbf{i} + \mathbf{j} \Rightarrow \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 0 \\ 3 & 1 & 0 \end{vmatrix} = -11\mathbf{k} \Rightarrow \text{area} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{11}{2}$$

$$54. \vec{AB} = 4\mathbf{i} + 4\mathbf{j} \text{ and } \vec{AC} = 3\mathbf{i} + 2\mathbf{j} \Rightarrow \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 4 & 0 \\ 3 & 2 & 0 \end{vmatrix} = -4\mathbf{k} \Rightarrow \text{area} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = 2$$

$$55. \vec{AB} = 6\mathbf{i} - 5\mathbf{j} \text{ and } \vec{AC} = 11\mathbf{i} - 5\mathbf{j} \Rightarrow \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -5 & 0 \\ 11 & -5 & 0 \end{vmatrix} = 25\mathbf{k} \Rightarrow \text{area} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{25}{2}$$

$$56. \vec{AB} = 16\mathbf{i} - 5\mathbf{j} \text{ and } \vec{AC} = 4\mathbf{i} + 4\mathbf{j} \Rightarrow \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 16 & -5 & 0 \\ 4 & 4 & 0 \end{vmatrix} = 84\mathbf{k} \Rightarrow \text{area} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = 42$$

$$57. \text{ If } \mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} \text{ and } \mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j}, \text{ then } \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \text{ and the triangle's area is}$$

$\frac{1}{2} |\mathbf{u} \times \mathbf{v}| = \pm \frac{1}{2} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$. The applicable sign is (+) if the acute angle from $\mathbf{u}$ to $\mathbf{v}$ runs counterclockwise in the xy -plane, and (−) if it runs clockwise, because the area must be a nonnegative number.

58. If $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j}$, $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j}$, and $\mathbf{w} = c_1\mathbf{i} + c_2\mathbf{j}$, then the area of the triangle is $\frac{1}{2} |\vec{AB} \times \vec{AC}|$. Now,

$$\begin{aligned} \vec{AB} \times \vec{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 - a_1 & b_2 - a_2 & 0 \\ c_1 - a_1 & c_2 - a_2 & 0 \end{vmatrix} = \begin{vmatrix} b_1 - a_1 & b_2 - a_2 \\ c_1 - a_1 & c_2 - a_2 \end{vmatrix} \mathbf{k} \Rightarrow \frac{1}{2} |\vec{AB} \times \vec{AC}| \\ &= \frac{1}{2} |(b_1 - a_1)(c_2 - a_2) - (c_1 - a_1)(b_2 - a_2)| = \frac{1}{2} |a_1(b_2 - c_2) + a_2(c_1 - b_1) + (b_1c_2 - c_1b_2)| \\ &= \pm \frac{1}{2} \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}. \text{ The applicable sign ensures the area formula gives a nonnegative number.} \end{aligned}$$

10.3 LINES AND PLANES IN SPACE

1. Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (3\mathbf{i} - 4\mathbf{j} - \mathbf{k}) + t(\mathbf{i} + \mathbf{j} + \mathbf{k}) = (3 + t)\mathbf{i} + (-4 + t)\mathbf{j} + (-1 + t)\mathbf{k}$
Parametric form: $x = 3 + t$, $y = -4 + t$, $z = -1 + t$

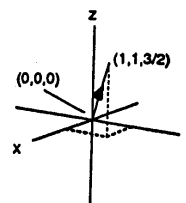
2. Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + t(-2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = (1 - 2t)\mathbf{i} + (2 - 2t)\mathbf{j} + (-1 + 2t)\mathbf{k}$
 Parametric form: $x = 1 - 2t, y = 2 - 2t, z = -1 + 2t$
3. Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (-2\mathbf{i} + 0\mathbf{j} + 3\mathbf{k}) + t(5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}) = (-2 + 5t)\mathbf{i} + (5t)\mathbf{j} + (3 - 5t)\mathbf{k}$
 Parametric form: $x = -2 + 5t, y = 5t, z = 3 - 5t$
4. Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) + t(2\mathbf{j} + \mathbf{k}) = 0\mathbf{i} + (2t)\mathbf{j} + (t)\mathbf{k}$
 Parametric form: $x = 0, y = 2t, z = t$
5. Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + t(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = (3 + 2t)\mathbf{i} + (-2 - t)\mathbf{j} + (1 + 3t)\mathbf{k}$
 Parametric form: $x = (3 + 2t), y = -2 - t, z = 1 + 3t$
6. Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) + t(0\mathbf{i} + 0\mathbf{j} + \mathbf{k}) = \mathbf{i} + \mathbf{j} + (1 + t)\mathbf{k}$
 Parametric form: $x = 1, y = 1, z = 1 + t$
7. Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (2\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) + t(3\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}) = (2 + 3t)\mathbf{i} + (4 + 7t)\mathbf{j} + (5 - 5t)\mathbf{k}$
 Parametric form: $x = 2 + 3t, y = 4 + 7t, z = 5 - 5t$
8. Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (0\mathbf{i} - 7\mathbf{j} + 0\mathbf{k}) + t(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = (t)\mathbf{i} + (-7 + 2t)\mathbf{j} + (2t)\mathbf{k}$
 Parametric form: $x = t, y = -7 + 2t, z = 2t$

9. The vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u}$ and $\mathbf{v}$: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = -2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

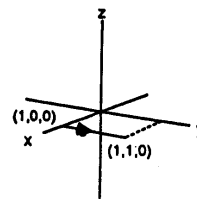
Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t(\mathbf{u} \times \mathbf{v}) = (2\mathbf{i} + 3\mathbf{j} + 0\mathbf{k}) + t(-2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = (2 - 2t)\mathbf{i} + (3 + 4t)\mathbf{j} + (-2t)\mathbf{k}$
 Parametric form: $x = 2 - 2t, y = 3 + 4t, z = -2t$

10. Vector form: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) + t(\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) = t\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = t\mathbf{i}$
 Parametric form: $x = t, y = 0, z = 0$

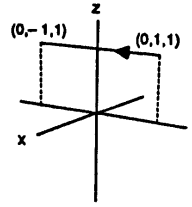
11. The direction $\vec{PQ} = \mathbf{i} + \mathbf{j} + \frac{3}{2}\mathbf{k}$ and $P(0, 0, 0) \Rightarrow x = t, y = t, z = \frac{3}{2}t$,
 where $0 \leq t \leq 1$



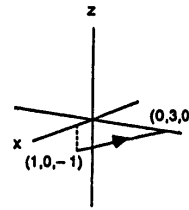
12. The direction $\vec{PQ} = \mathbf{j}$ and $P(1, 1, 0) \Rightarrow x = 1, y = 1 + t, z = 0$,
 where $-1 \leq t \leq 0$



13. The direction $\vec{PQ} = -2\mathbf{j}$ and $P(0, 1, 1) \Rightarrow x = 0, y = 1 - 2t, z = 1$,
where $0 \leq t \leq 1$



14. The direction $\vec{PQ} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $P(1, 0, -1) \Rightarrow x = 1 - t$,
 $y = 3t, z = -1 + t$, where $0 \leq t \leq 1$



15. $3(x - 0) + (-2)(y - 2) + (-1)(z + 1) = 0 \Rightarrow 3x - 2y - z = -3$

16. $3(x - 1) + (1)(y + 1) + (1)(z - 3) = 0 \Rightarrow 3x + y + z = 5$

17. $\vec{PQ} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}, \vec{PS} = -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \Rightarrow \vec{PQ} \times \vec{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = 7\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$ is normal to the plane

$\Rightarrow 7(x - 2) + (-5)(y - 0) + (-4)(z - 2) = 0 \Rightarrow 7x - 5y - 4z = 6$

18. $\vec{PQ} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}, \vec{PS} = -3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \Rightarrow \vec{PQ} \times \vec{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ -3 & 2 & 3 \end{vmatrix} = -\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is normal to the plane

$\Rightarrow (-1)(x - 1) + (-3)(y - 5) + (1)(z - 7) = 0 \Rightarrow x + 3y - z = 9$

19. $\mathbf{n} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}, P(2, 4, 5) = (1)(x - 2) + (3)(y - 4) + (4)(z - 5) = 0 \Rightarrow x + 3y + 4z = 34$

20. $\mathbf{n} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}, P(1, -2, 1) = (1)(x - 1) + (-2)(y + 2) + (1)(z - 1) = 0 \Rightarrow x - 2y + z = 6$

21. $\begin{cases} x = 2t + 1 = s + 2 \\ y = 3t + 2 = 2s + 4 \end{cases} \Rightarrow \begin{cases} 2t - s = 1 \\ 3t - 2s = 2 \end{cases} \Rightarrow \begin{cases} 4t - 2s = 2 \\ 3t - 2s = 2 \end{cases} \Rightarrow t = 0 \text{ and } s = -1; \text{ then } z = 4t + 3 = -4s - 1$

$\Rightarrow 4(0) + 3 = (-4)(-1) - 1$ is satisfied $\Rightarrow$ the lines do intersect when $t = 0$ and $s = -1 \Rightarrow$ the point of intersection is $x = 1, y = 2$, and $z = 3$ or $P(1, 2, 3)$. A vector normal to the plane determined by these lines is

$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix} = -20\mathbf{i} + 12\mathbf{j} + \mathbf{k}$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are directions of the lines $\Rightarrow$ the plane

containing the lines is represented by $(-20)(x-1) + (12)(y-2) + (1)(z-3) = 0 \Rightarrow -20x + 12y + z = 7$.

$$22. \begin{cases} x = t = 2s + 2 \\ y = -t + 2 = s + 3 \end{cases} \Rightarrow \begin{cases} t - 2s = 2 \\ -t - s = 1 \end{cases} \Rightarrow s = -1 \text{ and } t = 0; \text{ then } z = t + 1 = 5s + 6 \Rightarrow 0 + 1 = 5(-1) + 6$$

is satisfied $\Rightarrow$ the lines do intersect when $s = -1$ and $t = 0 \Rightarrow$ the point of intersection is $x = 0$, $y = 2$ and $z = 1$

or $P(0, 2, 1)$. A vector normal to the plane determined by these lines is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & 5 \end{vmatrix}$

$= -6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are directions of the lines $\Rightarrow$ the plane containing the lines is represented by $(-6)(x-0) + (-3)(y-2) + (3)(z-1) = 0 \Rightarrow 6x + 3y - 3z = 3$.

23. The cross product of $\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $-4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ has the same direction as the normal to the plane

$$\Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{vmatrix} = 6\mathbf{j} + 6\mathbf{k}. \text{ Select a point on either line, such as } P(-1, 2, 1). \text{ Since the lines are given}$$

to intersect, the desired plane is $0(x+1) + 6(y-2) + 6(z-1) = 0 \Rightarrow 6y + 6z = 18 \Rightarrow y + z = 3$.

24. The cross product of $\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + \mathbf{k}$ has the same direction as the normal to the plane

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}. \text{ Select a point on either line, such as } P(0, 3, -2). \text{ Since the lines are}$$

given to intersect, the desired plane is $(-2)(x-0) + (-2)(y-3) + (4)(z+2) = 0 \Rightarrow -2x - 2y + 4z = -14$
 $\Rightarrow x + y - 2z = 7$.

$$25. \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} \text{ is a vector in the direction of the line of intersection of the planes}$$

$\Rightarrow 3(x-2) + (-3)(y-1) + 3(z+1) = 0 \Rightarrow 3x - 3y + 3z = 0 \Rightarrow x - y + z = 0$ is the desired plane containing $P_0(2, 1, -1)$

$$26. \text{ A vector normal to the desired plane is } \vec{P_1P_2} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -2 \\ 4 & -1 & 2 \end{vmatrix} = -2\mathbf{i} - 12\mathbf{j} - 2\mathbf{k}; \text{ choosing } P_1(1, 2, 3) \text{ as a}$$

point on the plane $\Rightarrow (-2)(x-1) + (-12)(y-2) + (-2)(z-3) = 0 \Rightarrow -2x - 12y - 2z = -32 \Rightarrow x + 6y + z = 16$

is the desired plane

27. (a) If $\theta = 0$ or $\theta = \pi$, then S is on the line through P in the direction of $\mathbf{v}$. In these cases, the distance from the point S to the line is 0 and $|\vec{PS}| \sin \theta = 0$. If $0 < \theta < \frac{\pi}{2}$, then by the trigonometry of a right triangle, the distance from the point S to the line is $|\vec{PS}| \sin \theta$. If $\theta = \frac{\pi}{2}$ then $|\vec{PS}| \sin \theta = |\vec{PS}|$, which is the distance from S to the line. If $\frac{\pi}{2} < \theta < \pi$ then the distance from S to the line is $|\vec{PS}| \sin(\pi - \theta) = |\vec{PS}| \sin \theta$. Therefore, for all $0 \leq \theta \leq \pi$, the distance from the point S to the line through P and in the direction of $\mathbf{v}$ is given by $|\vec{PS}| \sin \theta$.

$$(b) |\vec{PS} \times \mathbf{v}| = |\vec{PS}| |\mathbf{v}| \sin \theta \Rightarrow \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} = |\vec{PS}| \sin \theta = d$$

$$28. S(0, 0, 0), P(5, 5, -3) \text{ and } \mathbf{v} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} \Rightarrow \vec{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & -5 & 3 \\ 3 & 4 & -5 \end{vmatrix} = 13\mathbf{i} + 16\mathbf{j} - 5\mathbf{k}$$

$$\Rightarrow d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{169 + 256 + 25}}{\sqrt{9 + 16 + 25}} = \frac{\sqrt{450}}{\sqrt{50}} = \sqrt{9} = 3 \text{ is the distance from S to the line}$$

$$29. S(2, 1, 3), P(2, 1, 3) \text{ and } \mathbf{v} = 2\mathbf{i} + 6\mathbf{j} \Rightarrow \vec{PS} \times \mathbf{v} = \mathbf{0} \Rightarrow d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{0}{\sqrt{40}} = 0 \text{ is the distance from S to the line}$$

(i.e., the point S lies on the line)

$$30. S(3, -1, 4), P(4, 3, -5) \text{ and } \mathbf{v} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \Rightarrow \vec{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -4 & 9 \\ -1 & 2 & 3 \end{vmatrix} = -30\mathbf{i} - 6\mathbf{j} - 6\mathbf{k}$$

$$\Rightarrow d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{900 + 36 + 36}}{\sqrt{1 + 4 + 9}} = \frac{\sqrt{972}}{\sqrt{14}} = \frac{\sqrt{486}}{\sqrt{7}} = \frac{\sqrt{81 \cdot 6}}{\sqrt{7}} = \frac{9\sqrt{42}}{7} \text{ is the distance from S to the line}$$

31. (a) P is any point with coordinates (x_0, y_0, z_0) that satisfies the equation $Ax_0 + By_0 + Cz_0 = D$.

(b) If S has coordinates (x, y, z) , then $\vec{PS} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$

(c) If the angle between $\vec{PS}$ and the normal to the plane is θ , then $d = |\vec{PS}| |\cos \theta|$. We know, however, that $\vec{PS} \cdot \mathbf{n} = |\vec{PS}| |\mathbf{n}| \cos \theta$, where $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane. This implies that

$$|\vec{PS}| \cos \theta = \frac{\vec{PS} \cdot \mathbf{n}}{|\mathbf{n}|} = \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \text{ and } ||\vec{PS}| \cos \theta| = |\vec{PS}| |\cos \theta| = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right|, \text{ and that } d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right|.$$

32. $S(2, -3, 4)$, $x + 2y + 2z = 13$ and $P(13, 0, 0)$ is on the plane $\Rightarrow \vec{PS} = -11\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

$$\Rightarrow d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = \left| \frac{-11 - 6 + 8}{\sqrt{1 + 4 + 4}} \right| = \left| \frac{-9}{\sqrt{9}} \right| = 3$$

33. $S(0, 1, 1)$, $4y + 3z = -12$ and $P(0, -3, 0)$ is on the plane $\Rightarrow \vec{PS} = 4\mathbf{j} + \mathbf{k}$ and $\mathbf{n} = 4\mathbf{j} + 3\mathbf{k}$

$$\Rightarrow d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = \left| \frac{16 + 3}{\sqrt{16 + 9}} \right| = \frac{19}{5}$$

34. $S(0, -1, 0)$, $2x + y + 2z = 4$ and $P(2, 0, 0)$ is on the plane $\Rightarrow \vec{PS} = -2\mathbf{i} - \mathbf{j}$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$$\Rightarrow d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = \left| \frac{-4 - 1 + 0}{\sqrt{4 + 1 + 4}} \right| = \frac{5}{3}$$

35. The point $P(1, 0, 0)$ is on the first plane and $S(10, 0, 0)$ is a point on the second plane $\Rightarrow \vec{PS} = 9\mathbf{i}$, and

$$\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k} \text{ is normal to the first plane } \Rightarrow \text{the distance from } S \text{ to the first plane is } d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right|$$

$$= \left| \frac{9}{\sqrt{1 + 4 + 36}} \right| = \frac{9}{\sqrt{41}}, \text{ which is also the distance between the planes.}$$

36. The line is parallel to the plane since $\mathbf{v} \cdot \mathbf{n} = (\mathbf{i} + \mathbf{j} - \frac{1}{2}\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}) = 1 + 2 - 3 = 0$. Also the point $S(1, 0, 0)$ when $t = -1$ lies on the line, and the point $P(10, 0, 0)$ lies on the plane $\Rightarrow \vec{PS} = -9\mathbf{i}$. The distance

$$\text{from } S \text{ to the plane is } d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = \left| \frac{-9}{\sqrt{1 + 4 + 36}} \right| = \frac{9}{\sqrt{41}}, \text{ which is also the distance from the line to the plane.}$$

37. (a) If two planes intersect, then the angle between the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ is equal to the angle between the planes. We know, however, that $\mathbf{n}_1 \cdot \mathbf{n}_2 = |\mathbf{n}_1| |\mathbf{n}_2| \cos \theta$, where θ is the angle between the normal

vectors and between the planes. Therefore, $\theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right)$ with $0 \leq \theta \leq \pi$.

$$(b) \mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k} \text{ and } \mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \theta = \cos^{-1} \left(\frac{6 - 6 + 4}{\sqrt{3^2 + 6^2 + 2^2} \sqrt{2^2 + 1^2 + 2^2}} \right) = \cos^{-1} \left(\frac{4}{21} \right)$$

$$\approx 1.38 \text{ radians.}$$

$$38. \mathbf{n}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k} \text{ and } \mathbf{n}_2 = \mathbf{k} \Rightarrow \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{1}{\sqrt{3} \sqrt{1}} \right) \approx 0.96 \text{ rad}$$

$$39. \mathbf{n}_1 = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k} \text{ and } \mathbf{n}_2 = \mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{2 + 4 - 1}{\sqrt{9} \sqrt{6}} \right) = \cos^{-1} \left(\frac{5}{3\sqrt{6}} \right) \approx 0.82 \text{ rad}$$

$$40. \mathbf{n}_1 = 4\mathbf{j} + 3\mathbf{k} \text{ and } \mathbf{n}_2 = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} \Rightarrow \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{8 + 18}{\sqrt{25} \sqrt{49}} \right) = \cos^{-1} \left(\frac{26}{35} \right) \approx 0.73 \text{ rad}$$

$$41. 2x - y + 3z = 6 \Rightarrow 2(1 - t) - (3t) + 3(1 + t) = 6 \Rightarrow -2t + 5 = 6 \Rightarrow t = -\frac{1}{2} \Rightarrow x = \frac{3}{2}, y = -\frac{3}{2} \text{ and } z = \frac{1}{2}$$

$$\Rightarrow \left(\frac{3}{2}, -\frac{3}{2}, \frac{1}{2} \right) \text{ is the point}$$

$$42. 6x + 3y - 4z = -12 \Rightarrow 6(2) + 3(3 + 2t) - 4(-2 - 2t) = -12 \Rightarrow 14t + 29 = -12 \Rightarrow t = -\frac{41}{14} \Rightarrow x = 2, y = 3 - \frac{41}{7},$$

$$\text{and } z = -2 + \frac{41}{7} \Rightarrow \left(2, -\frac{20}{7}, \frac{27}{7} \right) \text{ is the point}$$

43. $x + y + z = 2 \Rightarrow (1 + 2t) + (1 + 5t) + (3t) = 2 \Rightarrow 10t + 2 = 2 \Rightarrow t = 0 \Rightarrow x = 1, y = 1$ and $z = 0$
 $\Rightarrow (1, 1, 0)$ is the point

44. $2x - 3z = 7 \Rightarrow 2(-1 + 3t) - 3(5t) = 7 \Rightarrow -9t - 2 = 7 \Rightarrow t = -1 \Rightarrow x = -1 - 3, y = -2$ and $z = -5$
 $\Rightarrow (-4, -2, -5)$ is the point

45. $\mathbf{n}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{j}$, the direction of the desired line; $(1, 1, -1)$

is on both planes $\Rightarrow$ the desired line is $x = 1 - t, y = 1 + t, z = -1$

46. $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$, the direction of the

desired line; $(1, 0, 0)$ is on both planes $\Rightarrow$ the desired line is $x = 1 + 14t, y = 2t, z = 15t$

47. $\mathbf{n}_1 = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 4 \\ 1 & 1 & -2 \end{vmatrix} = 6\mathbf{j} + 3\mathbf{k}$, the direction of the

desired line; $(4, 3, 1)$ is on both planes $\Rightarrow$ the desired line is $x = 4, y = 3 + 6t, z = 1 + 3t$

48. $\mathbf{n}_1 = 5\mathbf{i} - 2\mathbf{j}$ and $\mathbf{n}_2 = 4\mathbf{j} - 5\mathbf{k} \Rightarrow \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -2 & 0 \\ 0 & 4 & -5 \end{vmatrix} = 10\mathbf{i} + 25\mathbf{j} + 20\mathbf{k}$, the direction of the

desired line; $(1, -3, 1)$ is on both planes $\Rightarrow$ the desired line is $x = 1 + 10t, y = -3 + 25t, z = 1 + 20t$

49. L1 & L2: $x = 3 + 2t = 1 + 4s$ and $y = -1 + 4t = 1 + 2s \Rightarrow \begin{cases} 2t - 4s = -2 \\ 4t - 2s = 2 \end{cases} \Rightarrow \begin{cases} 2t - 4s = -2 \\ 2t - s = 1 \end{cases}$
 $\Rightarrow -3s = -3 \Rightarrow s = 1$ and $t = 1 \Rightarrow$ on L1, $z = 1$ and on L2, $z = 1 \Rightarrow$ L1 and L2 intersect at $(5, 3, 1)$.

L2 & L3: The direction of L2 is $\frac{1}{6}(4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) = \frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ which is the same as the direction $\frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ of L3; hence L2 and L3 are parallel.

L1 & L3: $x = 3 + 2t = 3 + 2r$ and $y = -1 + 4t = 2 + r \Rightarrow \begin{cases} 2t - 2r = 0 \\ 4t - r = 3 \end{cases} \Rightarrow \begin{cases} t - r = 0 \\ 4t - r = 3 \end{cases} \Rightarrow 3t = 3$

$\Rightarrow t = 1$ and $r = 1 \Rightarrow$ on L1, $z = 2$ while on L3, $z = 0 \Rightarrow$ L1 and L2 do not intersect. The direction of L1 is $\frac{1}{\sqrt{21}}(2\mathbf{i} + 4\mathbf{j} - \mathbf{k})$ while the direction of L3 is $\frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and neither is a multiple of the other; hence

L1 and L3 are skew.

50. L1 & L2: $x = 1 + 2t = 2 - s$ and $y = -1 - t = 3s \Rightarrow \begin{cases} 2t + s = 1 \\ -t - 3s = 1 \end{cases} \Rightarrow -5s = 3 \Rightarrow s = -\frac{3}{5}$ and $t = \frac{4}{5} \Rightarrow$ on L1,

$z = \frac{12}{5}$ while on L2, $z = 1 - \frac{3}{5} = \frac{2}{5} \Rightarrow$ L1 and L2 do not intersect. The direction of L1 is $\frac{1}{\sqrt{14}}(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$

while the direction of L2 is $\frac{1}{\sqrt{11}}(-\mathbf{i} + 3\mathbf{j} + \mathbf{k})$ and neither is a multiple of the other; hence, L1 and L2 are skew.

L2 & L3: $x = 2 - s = 5 + 2r$ and $y = 3s = 1 - r \Rightarrow \begin{cases} -s - 2r = 3 \\ 3s + r = 1 \end{cases} \Rightarrow 5s = 5 \Rightarrow s = 1$ and $r = -2 \Rightarrow$ on L2,

$z = 2$ and on L3, $z = 2 \Rightarrow$ L2 and L3 intersect at $(1, 3, 2)$.

L1 & L3: L1 and L3 have the same direction $\frac{1}{\sqrt{14}}(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$; hence L1 and L3 are parallel.

51. $x = 2 + 2t$, $y = -4 - t$, $z = 7 + 3t$; $x = -2 - t$, $y = -2 + \frac{1}{2}t$, $z = 1 - \frac{3}{2}t$

52. $1(x - 4) - 2(y - 1) + 1(z - 5) = 0 \Rightarrow x - 4 - 2y + 2 + z - 5 = 0 \Rightarrow x - 2y + z = 7$;
 $-\sqrt{2}(x - 3) + 2\sqrt{2}(y + 2) - \sqrt{2}(z - 0) = 0 \Rightarrow -\sqrt{2}x + 2\sqrt{2}y - \sqrt{2}z = -7\sqrt{2}$

53. $x = 0 \Rightarrow t = -\frac{1}{2}$, $y = -\frac{1}{2}$, $z = -\frac{3}{2} \Rightarrow (0, -\frac{1}{2}, -\frac{3}{2})$; $y = 0 \Rightarrow t = -1$, $x = -1$, $z = -3 \Rightarrow (-1, 0, -3)$; $z = 0 \Rightarrow t = 0$, $x = 1$, $y = -1 \Rightarrow (1, -1, 0)$

54. The line contains $(0, 0, 3)$ and $(\sqrt{3}, 1, 3)$ because the projection of the line onto the xy -plane contains the origin and intersects the positive x -axis at a 30° angle. The direction of the line is $\sqrt{3}\mathbf{i} + \mathbf{j} + 0\mathbf{k} \Rightarrow$ the line in question is $x = \sqrt{3}t$, $y = t$, $z = 3$.

55. With substitution of the line into the plane we have $2(1 - 2t) + (2 + 5t) - (-3t) = 8 \Rightarrow 2 - 4t + 2 + 5t + 3t = 8 \Rightarrow 4t + 4 = 8 \Rightarrow t = 1 \Rightarrow$ the point $(-1, 7, -3)$ is contained in both the line and plane, so they are not parallel.

56. The planes are parallel when either vector $A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k}$ or $A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k}$ is a multiple of the other or when $|(A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k}) \times (A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k})| = 0$. The planes are perpendicular when their normals are perpendicular, or $(A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k}) \cdot (A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k}) = 0$.

57. There are many possible answers. One is found as follows: eliminate t to get $t = x - 1 = 2 - y = \frac{z - 3}{2} \Rightarrow x - 1 = 2 - y$ and $2 - y = \frac{z - 3}{2} \Rightarrow x + y = 3$ and $2y + z = 7$ are two such planes.

58. Since the plane passes through the origin, its general equation is of the form $Ax + By + Cz = 0$. Since it meets the plane M at a right angle, their normal vectors are perpendicular $\Rightarrow 2A + 3B + C = 0$. One choice satisfying this equation is $A = 1$, $B = -1$ and $C = 1 \Rightarrow x - y + z = 0$. Any plane $Ax + By + Cz = 0$ with $2A + 3B + C = 0$ will pass through the origin and be perpendicular to M .

59. The points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ are the x , y , and z intercepts of the plane. Since a , b , and c are all nonzero, the plane must intersect all three coordinate axes and cannot pass through the origin. Thus, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ describes all planes except those through the origin or parallel to a coordinate axis.

60. Yes. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero vectors parallel to the lines, then $\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$ is perpendicular to the lines.

61. (a) $\vec{EP} = c\vec{EP}_1 \Rightarrow -x_0\mathbf{i} + y\mathbf{j} + z\mathbf{k} = c[(x_1 - x_0)\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}] \Rightarrow -x_0 = c(x_1 - x_0)$, $y = cy_1$ and $z = cz_1$, where c is a positive real number

(b) At $x_1 = 0 \Rightarrow c = 1 \Rightarrow y = y_1$ and $z = z_1$; at $x_1 = x_0 \Rightarrow x_0 = 0$, $y = 0$, $z = 0$; $\lim_{x_0 \rightarrow \infty} c = \lim_{x_0 \rightarrow \infty} \frac{-x_0}{x_1 - x_0}$
 $= \lim_{x_0 \rightarrow \infty} \frac{-1}{-1} = 1 \Rightarrow c \rightarrow 1$ so that $y \rightarrow y_1$ and $z \rightarrow z_1$

62. The plane which contains the triangular plane is $x + y + z = 2$. The line containing the endpoints of the line segment is $x = 1 - t$, $y = 2t$, $z = 2t$. The plane and the line intersect at $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$. The visible section of the line segment is $\sqrt{(\frac{1}{3})^2 + (\frac{2}{3})^2 + (\frac{2}{3})^2} = 1$ unit in length. The length of the line segment is $\sqrt{1^2 + 2^2 + 2^2} = 3 \Rightarrow \frac{2}{3}$ of the line segment is hidden from view.

10.4 CYLINDERS AND QUADRIC SURFACES

- | | | |
|-------------------|-----------------------------|-----------------------------|
| 1. d, ellipsoid | 2. i, hyperboloid | 3. a, cylinder |
| 4. g, cone | 5. l, hyperbolic paraboloid | 6. e, paraboloid |
| 7. b, cylinder | 8. j, hyperboloid | 9. k, hyperbolic paraboloid |
| 10. f, paraboloid | 11. h, cone | 12. c, ellipsoid |

13. (a) If $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ and $z = c$, then $x^2 + \frac{y^2}{4} = \frac{9-c^2}{9} \Rightarrow \frac{x^2}{(\frac{9-c^2}{9})} + \frac{y^2}{4(\frac{9-c^2}{9})} = 1 \Rightarrow A = ab\pi$

$$= \pi \left(\frac{\sqrt{9-c^2}}{3} \right) \left(\frac{2\sqrt{9-c^2}}{3} \right) = \frac{2\pi(9-c^2)}{9}$$

(b) From part (a), each slice has the area $\frac{2\pi(9-z^2)}{9}$, where $-3 \leq z \leq 3$. Thus $V = 2 \int_0^3 \frac{2\pi(9-z^2)}{9} dz$

$$= \frac{4\pi}{9} \int_0^3 (9-z^2) dz = \frac{4\pi}{9} \left[9z - \frac{z^3}{3} \right]_0^3 = \frac{4\pi}{9} (27-9) = 8\pi$$

(c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow \frac{x^2}{\frac{a^2(c^2-z^2)}{c^2}} + \frac{y^2}{\frac{b^2(c^2-z^2)}{c^2}} = 1 \Rightarrow A = \pi \left(\frac{a\sqrt{c^2-z^2}}{c} \right) \left(\frac{b\sqrt{c^2-z^2}}{c} \right)$

$$\Rightarrow V = 2 \int_0^c \frac{\pi ab}{c^2} (c^2 - z^2) dz = \frac{2\pi ab}{c^2} \left[c^2 z - \frac{z^3}{3} \right]_0^c = \frac{2\pi ab}{c^2} \left(\frac{2}{3} c^3 \right) = \frac{4\pi abc}{3}. \text{ Note that if } r = a = b = c,$$

then $V = \frac{4\pi r^3}{3}$, which is the volume of a sphere.

14. The ellipsoid has the form $\frac{x^2}{R^2} + \frac{y^2}{R^2} + \frac{z^2}{c^2} = 1$. To determine c^2 we note that the point $(0, r, h)$ lies on the surface

of the barrel. Thus, $\frac{r^2}{R^2} + \frac{h^2}{c^2} = 1 \Rightarrow c^2 = \frac{h^2 R^2}{R^2 - r^2}$. We calculate the volume by the disk method:

$$\begin{aligned} V &= \pi \int_{-h}^h y^2 dz. \text{ Now, } \frac{y^2}{R^2} + \frac{z^2}{c^2} = 1 \Rightarrow y^2 = R^2 \left(1 - \frac{z^2}{c^2} \right) = R^2 \left[1 - \frac{z^2 (R^2 - r^2)}{h^2 R^2} \right] = R^2 - \left(\frac{R^2 - r^2}{h^2} \right) z^2 \\ \Rightarrow V &= \pi \int_{-h}^h \left[R^2 - \left(\frac{R^2 - r^2}{h^2} \right) z^2 \right] dz = \pi \left[R^2 z - \frac{1}{3} \left(\frac{R^2 - r^2}{h^2} \right) z^3 \right]_{-h}^h = 2\pi \left[R^2 h - \frac{1}{3} (R^2 - r^2) h \right] = 2\pi \left(\frac{2R^2 h}{3} + \frac{r^2 h}{3} \right) \\ &= \frac{4}{3} \pi R^2 h + \frac{2}{3} \pi r^2 h, \text{ the volume of the barrel. If } r = R, \text{ then } V = 2\pi R^2 h \text{ which is the volume of a cylinder of} \\ &\text{radius } R \text{ and height } 2h. \text{ If } r = 0 \text{ and } h = R, \text{ then } V = \frac{4}{3} \pi R^3 \text{ which is the volume of a sphere.} \end{aligned}$$

15. We calculate the volume by the slicing method, taking slices parallel to the xy -plane. For fixed z , $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$

gives the ellipse $\left(\frac{x}{\frac{a\sqrt{z}}{c}} \right)^2 + \left(\frac{y}{\frac{b\sqrt{z}}{c}} \right)^2 = 1$. The area of this ellipse is $\pi \left(a\sqrt{\frac{z}{c}} \right) \left(b\sqrt{\frac{z}{c}} \right) = \frac{\pi abz}{c}$ (see Exercise 13a). Hence

the volume is given by $V = \int_0^h \frac{\pi abz}{c} dz = \left[\frac{\pi abz^2}{2c} \right]_0^h = \frac{\pi abh^2}{2c}$. Now the area of the elliptic base when $z = h$ is

$A = \frac{\pi abh}{c}$, as determined previously. Thus, $V = \frac{\pi abh^2}{2c} = \frac{1}{2} \left(\frac{\pi abh}{c} \right) h = \frac{1}{2} (\text{base})(\text{altitude})$, as claimed.

16. (a) For each fixed value of z , the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ results in a cross-sectional ellipse

$$\left[\frac{x^2}{\frac{a^2(c^2 + z^2)}{c^2}} \right] + \left[\frac{y^2}{\frac{b^2(c^2 + z^2)}{c^2}} \right] = 1. \text{ The area of the cross-sectional ellipse (see Exercise 13a) is}$$

$$A(z) = \pi \left(\frac{a}{c} \sqrt{c^2 + z^2} \right) \left(\frac{b}{c} \sqrt{c^2 + z^2} \right) = \frac{\pi ab}{c^2} (c^2 + z^2). \text{ The volume of the solid by the method of slices is}$$

$$V = \int_0^h A(z) dz = \int_0^h \frac{\pi ab}{c^2} (c^2 + z^2) dz = \frac{\pi ab}{c^2} \left[c^2 z + \frac{1}{3} z^3 \right]_0^h = \frac{\pi ab}{c^2} \left(c^2 h + \frac{1}{3} h^3 \right) = \frac{\pi abh}{3c^2} (3c^2 + h^2)$$

$$(b) A_0 = A(0) = \pi ab \text{ and } A_h = A(h) = \frac{\pi ab}{c^2} (c^2 + h^2), \text{ from part (a)} \Rightarrow V = \frac{\pi abh}{3c^2} (3c^2 + h^2)$$

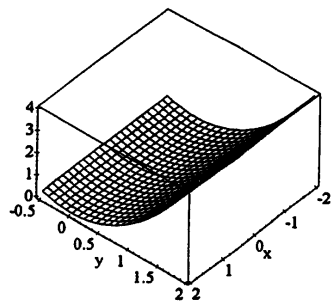
$$= \frac{\pi abh}{3} \left(2 + 1 + \frac{h^2}{c^2} \right) = \frac{\pi abh}{3} \left(2 + \frac{c^2 + h^2}{c^2} \right) = \frac{h}{3} \left[2\pi ab + \frac{\pi ab}{c^2} (c^2 + h^2) \right] = \frac{h}{3} (2A_0 + A_h)$$

$$(c) A_m = A\left(\frac{h}{2}\right) = \frac{\pi ab}{c^2} \left(c^2 + \frac{h^2}{4} \right) = \frac{\pi ab}{4c^2} (4c^2 + h^2) \Rightarrow \frac{h}{6} (A_0 + 4A_m + A_h)$$

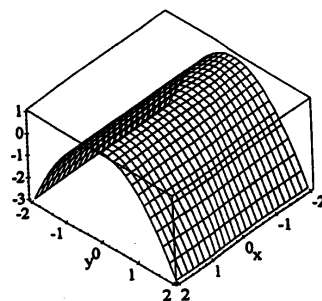
$$= \frac{h}{6} \left[\pi ab + \frac{\pi ab}{c^2} (4c^2 + h^2) + \frac{\pi ab}{c^2} (c^2 + h^2) \right] = \frac{\pi abh}{6c^2} (c^2 + 4c^2 + h^2 + c^2 + h^2) = \frac{\pi abh}{6c^2} (6c^2 + 2h^2)$$

$$= \frac{\pi abh}{3c^2} (3c^2 + h^2) = V \text{ from part (a)}$$

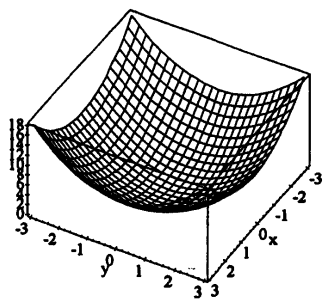
17. $z = y^2$



18. $z = 1 - y^2$

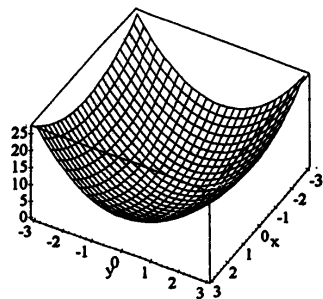


19. $z = x^2 + y^2$

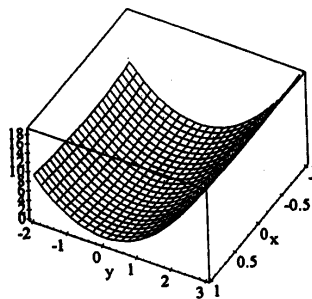


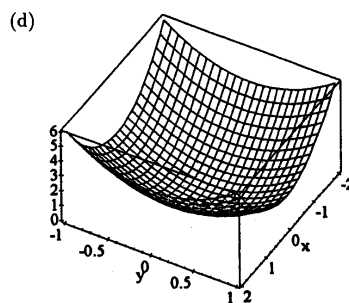
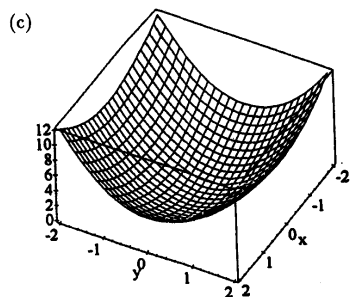
20. $z = x^2 + 2y^2$

(a)



(b)





21-26. Example CAS commands:

Maple:

```
with(plots):
eq1:= x^2/9 - y^2/16 - z^2/2 = 1;
implicitplot3d(eq1, x = -15..15, y = -9..9, z = -7..7, title = 'Hyperboloid of Two Sheets');
```

Mathematica:

```
ContourPlot3D[ x^2/9 - y^2/16 - z^2/2 - 1,
{x, -9, 9}, {y, -12, 12}, {z, -5, 5},
PlotLabel -> "Elliptic Hyperboloid of Two Sheets" ]
```

10.5 VECTOR-VALUED FUNCTIONS AND SPACE CURVES

1. $\mathbf{r} = (t+1)\mathbf{i} + (t^2-1)\mathbf{j} + 2t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = 2\mathbf{j}$; Speed: $|\mathbf{v}(1)| = \sqrt{1^2 + (2(1))^2 + 2^2} = 3$;

Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\mathbf{i} + 2(1)\mathbf{j} + 2\mathbf{k}}{3} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \Rightarrow \mathbf{v}(1) = 3\left(\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right)$

2. $\mathbf{r} = (1+t)\mathbf{i} + \frac{t^2}{\sqrt{2}}\mathbf{j} + \frac{t^3}{3}\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + \frac{2t}{\sqrt{2}}\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \frac{2}{\sqrt{2}}\mathbf{j} + 2t\mathbf{k}$; Speed: $|\mathbf{v}(1)|$

$$= \sqrt{1^2 + \left(\frac{2(1)}{\sqrt{2}}\right)^2 + (1^2)^2} = 2; \text{ Direction: } \frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\mathbf{i} + \frac{2(1)}{\sqrt{2}}\mathbf{j} + (1^2)\mathbf{k}}{2} = \frac{1}{2}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{2}\mathbf{k} \Rightarrow \mathbf{v}(1)$$

$$= 2\left(\frac{1}{2}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{2}\mathbf{k}\right)$$

3. $\mathbf{r} = (2 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (-2 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 4\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = (-2 \cos t)\mathbf{i} - (3 \sin t)\mathbf{j}$;

Speed: $|\mathbf{v}(\frac{\pi}{2})| = \sqrt{(-2 \sin \frac{\pi}{2})^2 + (3 \cos \frac{\pi}{2})^2 + 4^2} = 2\sqrt{5}$; Direction: $\frac{\mathbf{v}(\frac{\pi}{2})}{|\mathbf{v}(\frac{\pi}{2})|}$

$$= \left(-\frac{2}{2\sqrt{5}} \sin \frac{\pi}{2} \right) \mathbf{i} + \left(\frac{3}{2\sqrt{5}} \cos \frac{\pi}{2} \right) \mathbf{j} + \frac{4}{2\sqrt{5}} \mathbf{k} = -\frac{1}{\sqrt{5}} \mathbf{i} + \frac{2}{\sqrt{5}} \mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{2}\right) = 2\sqrt{5} \left(-\frac{1}{\sqrt{5}} \mathbf{i} + \frac{2}{\sqrt{5}} \mathbf{k} \right)$$

$$\begin{aligned} 4. \quad \mathbf{r} &= (\sec t) \mathbf{i} + (\tan t) \mathbf{j} + \frac{4}{3} t \mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (\sec t \tan t) \mathbf{i} + (\sec^2 t) \mathbf{j} + \frac{4}{3} \mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} \\ &= (\sec t \tan^2 t + \sec^3 t) \mathbf{i} + (2 \sec^2 t \tan t) \mathbf{j}; \text{ Speed: } \left| \mathbf{v}\left(\frac{\pi}{6}\right) \right| = \sqrt{\left(\sec \frac{\pi}{6} \tan \frac{\pi}{6} \right)^2 + \left(\sec^2 \frac{\pi}{6} \right)^2 + \left(\frac{4}{3} \right)^2} = 2; \\ \text{Direction: } \frac{\mathbf{v}\left(\frac{\pi}{6}\right)}{\left| \mathbf{v}\left(\frac{\pi}{6}\right) \right|} &= \frac{\left(\sec \frac{\pi}{6} \tan \frac{\pi}{6} \right) \mathbf{i} + \left(\sec^2 \frac{\pi}{6} \right) \mathbf{j} + \frac{4}{3} \mathbf{k}}{2} = \frac{1}{3} \mathbf{i} + \frac{2}{3} \mathbf{j} + \frac{2}{3} \mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{6}\right) = 2 \left(\frac{1}{3} \mathbf{i} + \frac{2}{3} \mathbf{j} + \frac{2}{3} \mathbf{k} \right) \end{aligned}$$

$$5. \quad \mathbf{r} = (2 \ln(t+1)) \mathbf{i} + t^2 \mathbf{j} + \frac{t^2}{2} \mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(\frac{2}{t+1} \right) \mathbf{i} + 2t \mathbf{j} + t \mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \left[\frac{-2}{(t+1)^2} \right] \mathbf{i} + 2 \mathbf{j} + \mathbf{k};$$

$$\begin{aligned} \text{Speed: } \left| \mathbf{v}(1) \right| &= \sqrt{\left(\frac{2}{1+1} \right)^2 + (2(1))^2 + 1^2} = \sqrt{6}; \text{ Direction: } \frac{\mathbf{v}(1)}{\left| \mathbf{v}(1) \right|} = \frac{\left(\frac{2}{1+1} \right) \mathbf{i} + 2(1) \mathbf{j} + (1) \mathbf{k}}{\sqrt{6}} \\ &= \frac{1}{\sqrt{6}} \mathbf{i} + \frac{2}{\sqrt{6}} \mathbf{j} + \frac{1}{\sqrt{6}} \mathbf{k} \Rightarrow \mathbf{v}(1) = \sqrt{6} \left(\frac{1}{\sqrt{6}} \mathbf{i} + \frac{2}{\sqrt{6}} \mathbf{j} + \frac{1}{\sqrt{6}} \mathbf{k} \right) \end{aligned}$$

$$\begin{aligned} 6. \quad \mathbf{r} &= (e^{-t}) \mathbf{i} + (2 \cos 3t) \mathbf{j} + (2 \sin 3t) \mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (-e^{-t}) \mathbf{i} - (6 \sin 3t) \mathbf{j} + (6 \cos 3t) \mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} \\ &= (-e^{-t}) \mathbf{i} - (18 \cos 3t) \mathbf{j} - (18 \sin 3t) \mathbf{k}; \text{ Speed: } \left| \mathbf{v}(0) \right| = \sqrt{(-e^0)^2 + [-6 \sin 3(0)]^2 + [6 \cos 3(0)]^2} = \sqrt{37}; \\ \text{Direction: } \frac{\mathbf{v}(0)}{\left| \mathbf{v}(0) \right|} &= \frac{(-e^0) \mathbf{i} - 6 \sin 3(0) \mathbf{j} + 6 \cos 3(0) \mathbf{k}}{\sqrt{37}} = -\frac{1}{\sqrt{37}} \mathbf{i} + \frac{6}{\sqrt{37}} \mathbf{k} \Rightarrow \mathbf{v}(0) = \sqrt{37} \left(-\frac{1}{\sqrt{37}} \mathbf{i} + \frac{6}{\sqrt{37}} \mathbf{k} \right) \end{aligned}$$

$$7. \quad \mathbf{v} = 3\mathbf{i} + \sqrt{3}\mathbf{j} + 2t\mathbf{k} \text{ and } \mathbf{a} = 2\mathbf{k} \Rightarrow \mathbf{v}(0) = 3\mathbf{i} + \sqrt{3}\mathbf{j} \text{ and } \mathbf{a}(0) = 2\mathbf{k} \Rightarrow \left| \mathbf{v}(0) \right| = \sqrt{3^2 + (\sqrt{3})^2 + 0^2} = \sqrt{12} \text{ and } \left| \mathbf{a}(0) \right| = \sqrt{2^2} = 2; \mathbf{v}(0) \cdot \mathbf{a}(0) = 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

$$\begin{aligned} 8. \quad \mathbf{v} &= \frac{\sqrt{2}}{2} \mathbf{i} + \left(\frac{\sqrt{2}}{2} - 32t \right) \mathbf{j} \text{ and } \mathbf{a} = -32\mathbf{j} \Rightarrow \mathbf{v}(0) = \frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j} \text{ and } \mathbf{a}(0) = -32\mathbf{j} \Rightarrow \left| \mathbf{v}(0) \right| = \sqrt{\left(\frac{\sqrt{2}}{2} \right)^2 + \left(\frac{\sqrt{2}}{2} \right)^2} \\ &= 1 \text{ and } \left| \mathbf{a}(0) \right| = \sqrt{(-32)^2} = 32; \mathbf{v}(0) \cdot \mathbf{a}(0) = \left(\frac{\sqrt{2}}{2} \right) (-32) = -16\sqrt{2} \Rightarrow \cos \theta = \frac{-16\sqrt{2}}{1(32)} = -\frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{3\pi}{4} \end{aligned}$$

$$\begin{aligned} 9. \quad \mathbf{v} &= \left(\frac{2t}{t^2+1} \right) \mathbf{i} + \left(\frac{1}{t^2+1} \right) \mathbf{j} + t(t^2+1)^{-1/2} \mathbf{k} \text{ and } \mathbf{a} = \left[\frac{-2t^2+2}{(t^2+1)^2} \right] \mathbf{i} - \left[\frac{2t}{(t^2+1)^2} \right] \mathbf{j} + \left[\frac{1}{(t^2+1)^{3/2}} \right] \mathbf{k} \Rightarrow \mathbf{v}(0) = \mathbf{j} \text{ and } \\ \mathbf{a}(0) &= 2\mathbf{i} + \mathbf{k} \Rightarrow \left| \mathbf{v}(0) \right| = 1 \text{ and } \left| \mathbf{a}(0) \right| = \sqrt{2^2 + 1^2} = \sqrt{5}; \mathbf{v}(0) \cdot \mathbf{a}(0) = 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} 10. \quad \mathbf{v} &= \frac{2}{3}(1+t)^{1/2} \mathbf{i} - \frac{2}{3}(1-t)^{1/2} \mathbf{j} + \frac{1}{3} \mathbf{k} \text{ and } \mathbf{a} = \frac{1}{3}(1+t)^{-1/2} \mathbf{i} + \frac{1}{3}(1-t)^{-1/2} \mathbf{j} \Rightarrow \mathbf{v}(0) = \frac{2}{3} \mathbf{i} - \frac{2}{3} \mathbf{j} + \frac{1}{3} \mathbf{k} \text{ and } \\ \mathbf{a}(0) &= \frac{1}{3} \mathbf{i} + \frac{1}{3} \mathbf{j} \Rightarrow \left| \mathbf{v}(0) \right| = \sqrt{\left(\frac{2}{3} \right)^2 + \left(-\frac{2}{3} \right)^2 + \left(\frac{1}{3} \right)^2} = 1 \text{ and } \left| \mathbf{a}(0) \right| = \sqrt{\left(\frac{1}{3} \right)^2 + \left(\frac{1}{3} \right)^2} = \frac{\sqrt{2}}{3}; \mathbf{v}(0) \cdot \mathbf{a}(0) = \frac{2}{9} - \frac{2}{9} \end{aligned}$$

$$= 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

$$11. \mathbf{v} = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j} \text{ and } \mathbf{a} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{v} \cdot \mathbf{a} = (\sin t)(1 - \cos t) + (\sin t)(\cos t) = \sin t. \text{ Thus, } \\ \mathbf{v} \cdot \mathbf{a} = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0, \pi, \text{ or } 2\pi$$

$$12. \mathbf{v} = (\cos t)\mathbf{i} + \mathbf{j} - (\sin t)\mathbf{k} \text{ and } \mathbf{a} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{k} \Rightarrow \mathbf{v} \cdot \mathbf{a} = -\sin t \cos t + \sin t \cos t = 0 \text{ for all } t \geq 0$$

$$13. \int_0^1 [t^3\mathbf{i} + 7\mathbf{j} + (t+1)\mathbf{k}] dt = \left[\frac{t^4}{4}\right]_0^1 \mathbf{i} + [7t]_0^1 \mathbf{j} + \left[\frac{t^2}{2} + t\right]_0^1 \mathbf{k} = \frac{1}{4}\mathbf{i} + 7\mathbf{j} + \frac{3}{2}\mathbf{k}$$

$$14. \int_1^2 \left[(6-6t)\mathbf{i} + 3\sqrt{t}\mathbf{j} + \left(\frac{4}{t^2}\right)\mathbf{k} \right] dt = [6t - 3t^2]_1^2 \mathbf{i} + [2t^{3/2}]_1^2 \mathbf{j} + [-4t^{-1}]_1^2 \mathbf{k} = -3\mathbf{i} + (4\sqrt{2} - 2)\mathbf{j} + 2\mathbf{k}$$

$$15. \int_{-\pi/4}^{\pi/4} [(\sin t)\mathbf{i} + (1 + \cos t)\mathbf{j} + (\sec^2 t)\mathbf{k}] dt = [-\cos t]_{-\pi/4}^{\pi/4} \mathbf{i} + [t + \sin t]_{-\pi/4}^{\pi/4} \mathbf{j} + [\tan t]_{-\pi/4}^{\pi/4} \mathbf{k} \\ = \left(\frac{\pi + 2\sqrt{2}}{2}\right)\mathbf{j} + 2\mathbf{k}$$

$$16. \int_0^{\pi/3} [(\sec t \tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2 \sin t \cos t)\mathbf{k}] dt = \int_0^{\pi/3} [(\sec t \tan t)\mathbf{i} + (\tan t)\mathbf{j} + (\sin 2t)\mathbf{k}] dt \\ = [\sec t]_0^{\pi/3} \mathbf{i} + [-\ln(\cos t)]_0^{\pi/3} \mathbf{j} + \left[-\frac{1}{2} \cos 2t\right]_0^{\pi/3} \mathbf{k} = \mathbf{i} + (\ln 2)\mathbf{j} + \frac{3}{4}\mathbf{k}$$

$$17. \int_1^4 \left(\frac{1}{t}\mathbf{i} + \frac{1}{5-t}\mathbf{j} + \frac{1}{2t}\mathbf{k} \right) dt = [\ln t]_1^4 \mathbf{i} + [-\ln(5-t)]_1^4 \mathbf{j} + \left[\frac{1}{2} \ln t\right]_1^4 \mathbf{k} = (\ln 4)\mathbf{i} + (\ln 4)\mathbf{j} + (\ln 2)\mathbf{k}$$

$$18. \int_0^1 \left(\frac{2}{\sqrt{1-t^2}}\mathbf{j} + \frac{\sqrt{3}}{1+t^2}\mathbf{k} \right) dt = [2 \sin^{-1} t]_0^1 \mathbf{i} + [\sqrt{3} \tan^{-1} t]_0^1 \mathbf{k} = \pi\mathbf{i} + \frac{\pi\sqrt{3}}{4}\mathbf{k}$$

$$19. \mathbf{r} = \int (-t\mathbf{i} - t\mathbf{j} - t\mathbf{k}) dt = -\frac{t^2}{2}\mathbf{i} - \frac{t^2}{2}\mathbf{j} - \frac{t^2}{2}\mathbf{k} + \mathbf{C}; \mathbf{r}(0) = 0\mathbf{i} - 0\mathbf{j} - 0\mathbf{k} + \mathbf{C} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \\ \Rightarrow \mathbf{r} = \left(-\frac{t^2}{2} + 1\right)\mathbf{i} + \left(-\frac{t^2}{2} + 2\right)\mathbf{j} + \left(-\frac{t^2}{2} + 3\right)\mathbf{k}$$

$$20. \mathbf{r} = \int [(180t)\mathbf{i} + (180t - 16t^2)\mathbf{j}] dt = 90t^2\mathbf{i} + \left(90t^2 - \frac{16}{3}t^3\right)\mathbf{j} + \mathbf{C}; \mathbf{r}(0) = 90(0)^2\mathbf{i} + \left[90(0)^2 - \frac{16}{3}(0)^3\right]\mathbf{j} + \mathbf{C} \\ = 100\mathbf{j} \Rightarrow \mathbf{C} = 100\mathbf{j} \Rightarrow \mathbf{r} = 90t^2\mathbf{i} + \left(90t^2 - \frac{16}{3}t^3 + 100\right)\mathbf{j}$$

$$21. \mathbf{r} = \int \left[\left(\frac{3}{2}(t+1)^{1/2} \right) \mathbf{i} + e^{-t} \mathbf{j} + \left(\frac{1}{t+1} \right) \mathbf{k} \right] dt = (t+1)^{3/2} \mathbf{i} - e^{-t} \mathbf{j} + \ln(t+1) \mathbf{k} + \mathbf{C};$$

$$\mathbf{r}(0) = (0+1)^{3/2} \mathbf{i} - e^{-0} \mathbf{j} + \ln(0+1) \mathbf{k} + \mathbf{C} = \mathbf{k} \Rightarrow \mathbf{C} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\Rightarrow \mathbf{r} = [(t+1)^{3/2} - 1] \mathbf{i} + (1 - e^{-t}) \mathbf{j} + [1 + \ln(t+1)] \mathbf{k}$$

$$22. \mathbf{r} = \int [(t^3 + 4t) \mathbf{i} + t \mathbf{j} + 2t^2 \mathbf{k}] dt = \left(\frac{t^4}{4} + 2t^2 \right) \mathbf{i} + \frac{t^2}{2} \mathbf{j} + \frac{2t^3}{3} \mathbf{k} + \mathbf{C}; \mathbf{r}(0) = \left[\frac{0^4}{4} + 2(0)^2 \right] \mathbf{i} + \frac{0^2}{2} \mathbf{j} + \frac{2(0)^3}{3} \mathbf{k} + \mathbf{C}$$

$$= \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{C} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{r} = \left(\frac{t^4}{4} + 2t^2 + 1 \right) \mathbf{i} + \left(\frac{t^2}{2} + 1 \right) \mathbf{j} + \frac{2t^3}{3} \mathbf{k}$$

$$23. \frac{d\mathbf{r}}{dt} = \int (-32\mathbf{k}) dt = -32t\mathbf{k} + \mathbf{C}_1; \frac{d\mathbf{r}}{dt}(0) = 8\mathbf{i} + 8\mathbf{j} \Rightarrow -32(0)\mathbf{k} + \mathbf{C}_1 = 8\mathbf{i} + 8\mathbf{j} \Rightarrow \mathbf{C}_1 = 8\mathbf{i} + 8\mathbf{j}$$

$$\Rightarrow \frac{d\mathbf{r}}{dt} = 8\mathbf{i} + 8\mathbf{j} - 32t\mathbf{k}; \mathbf{r} = \int (8\mathbf{i} + 8\mathbf{j} - 32t\mathbf{k}) dt = 8t\mathbf{i} + 8t\mathbf{j} - 16t^2\mathbf{k} + \mathbf{C}_2; \mathbf{r}(0) = 100\mathbf{k}$$

$$\Rightarrow 8(0)\mathbf{i} + 8(0)\mathbf{j} - 16(0)^2\mathbf{k} + \mathbf{C}_2 = 100\mathbf{k} \Rightarrow \mathbf{C}_2 = 100\mathbf{k} \Rightarrow \mathbf{r} = 8t\mathbf{i} + 8t\mathbf{j} + (100 - 16t^2)\mathbf{k}$$

$$24. \frac{d\mathbf{r}}{dt} = \int -(\mathbf{i} + \mathbf{j} + \mathbf{k}) dt = -(t\mathbf{i} + t\mathbf{j} + t\mathbf{k}) + \mathbf{C}_1; \frac{d\mathbf{r}}{dt}(0) = \mathbf{0} \Rightarrow -(0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) + \mathbf{C}_1 = \mathbf{0} \Rightarrow \mathbf{C}_1 = \mathbf{0}$$

$$\Rightarrow \frac{d\mathbf{r}}{dt} = -(t\mathbf{i} + t\mathbf{j} + t\mathbf{k}); \mathbf{r} = \int -(t\mathbf{i} + t\mathbf{j} + t\mathbf{k}) dt = -\left(\frac{t^2}{2} \mathbf{i} + \frac{t^2}{2} \mathbf{j} + \frac{t^2}{2} \mathbf{k} \right) + \mathbf{C}_2; \mathbf{r}(0) = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$$

$$\Rightarrow -\left(\frac{0^2}{2} \mathbf{i} + \frac{0^2}{2} \mathbf{j} + \frac{0^2}{2} \mathbf{k} \right) + \mathbf{C}_2 = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k} \Rightarrow \mathbf{C}_2 = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$$

$$\Rightarrow \mathbf{r} = \left(-\frac{t^2}{2} + 10 \right) \mathbf{i} + \left(-\frac{t^2}{2} + 10 \right) \mathbf{j} + \left(-\frac{t^2}{2} + 10 \right) \mathbf{k}$$

$$25. \mathbf{r}(t) = (\sin t)\mathbf{i} + (t^2 - \cos t)\mathbf{j} + e^t\mathbf{k} \Rightarrow \mathbf{v}(t) = (\cos t)\mathbf{i} + (2t + \sin t)\mathbf{j} + e^t\mathbf{k}; t_0 = 0 \Rightarrow \mathbf{v}(0) = \mathbf{i} + \mathbf{k} \text{ and}$$

$$\mathbf{r}(0) = \mathbf{P}_0 = (0, -1, 1) \Rightarrow x = 0 + t = t, y = -1, \text{ and } z = 1 + t \text{ are parametric equations of the tangent line}$$

$$26. \mathbf{r}(t) = (2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + 5t\mathbf{k} \Rightarrow \mathbf{v}(t) = (2 \cos t)\mathbf{i} - (2 \sin t)\mathbf{j} + 5\mathbf{k}; t_0 = 4\pi \Rightarrow \mathbf{v}(0) = 2\mathbf{i} + 5\mathbf{k} \text{ and}$$

$$\mathbf{r}(0) = \mathbf{P}_0 = (0, 2, 20\pi) \Rightarrow x = 0 + 2t = 2t, y = 2, \text{ and } z = 20\pi + 5t \text{ are parametric equations of the tangent line}$$

$$27. \mathbf{r}(t) = (a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + bt\mathbf{k} \Rightarrow \mathbf{v}(t) = (a \cos t)\mathbf{i} - (a \sin t)\mathbf{j} + b\mathbf{k}; t_0 = 2\pi \Rightarrow \mathbf{v}(0) = a\mathbf{i} + b\mathbf{k} \text{ and}$$

$$\mathbf{r}(0) = \mathbf{P}_0 = (0, a, 2b\pi) \Rightarrow x = 0 + at = at, y = a, \text{ and } z = 2\pi b + bt \text{ are parametric equations of the tangent line}$$

$$28. \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k} \Rightarrow \mathbf{v}(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (2 \cos 2t)\mathbf{k}; t_0 = \frac{\pi}{2} \Rightarrow \mathbf{v}(0) = -\mathbf{i} - 2\mathbf{k} \text{ and}$$

$$\mathbf{r}(0) = \mathbf{P}_0 = (0, 1, 0) \Rightarrow x = 0 - t = -t, y = 1, \text{ and } z = 0 - 2t = -2t \text{ are parametric equations of the tangent line}$$

$$29. \frac{d\mathbf{v}}{dt} = \mathbf{a} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}(t) = 3t\mathbf{i} - t\mathbf{j} + t\mathbf{k} + \mathbf{C}_1; \text{ the particle travels in the direction of the vector}$$

$$(4-1)\mathbf{i} + (1-2)\mathbf{j} + (4-3)\mathbf{k} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} \text{ (since it travels in a straight line), and at time } t = 0 \text{ it has speed}$$

$$2 \Rightarrow \mathbf{v}(0) = \frac{2}{\sqrt{9+1+1}}(3\mathbf{i} - \mathbf{j} + \mathbf{k}) = \mathbf{C}_1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{v}(t) = \left(3t + \frac{6}{\sqrt{11}} \right) \mathbf{i} - \left(t + \frac{2}{\sqrt{11}} \right) \mathbf{j} + \left(t + \frac{2}{\sqrt{11}} \right) \mathbf{k}$$

$$\Rightarrow \mathbf{r}(t) = \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t \right) \mathbf{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t \right) \mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t \right) \mathbf{k} + \mathbf{C}_2; \mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} = \mathbf{C}_2$$

$$\begin{aligned}\Rightarrow \mathbf{r}(t) &= \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t + 1\right)\mathbf{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t - 2\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t + 3\right)\mathbf{k} \\ &= \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)(3\mathbf{i} - \mathbf{j} + \mathbf{k}) + (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})\end{aligned}$$

30. $\frac{d\mathbf{v}}{dt} = \mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}(t) = 2t\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}_1$; the particle travels in the direction of the vector

$(3-1)\mathbf{i} + (0-(-1))\mathbf{j} + (3-2)\mathbf{k} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ (since it travels in a straight line), and at time $t = 0$ it has speed 2

$$\Rightarrow \mathbf{v}(0) = \frac{2}{\sqrt{4+1+1}}(2\mathbf{i} + \mathbf{j} + \mathbf{k}) = \mathbf{C}_1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{v}(t) = \left(2t + \frac{4}{\sqrt{6}}\right)\mathbf{i} + \left(t + \frac{2}{\sqrt{6}}\right)\mathbf{j} + \left(t + \frac{2}{\sqrt{6}}\right)\mathbf{k}$$

$$\Rightarrow \mathbf{r}(t) = \left(t^2 + \frac{4}{\sqrt{6}}t\right)\mathbf{i} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)\mathbf{k} + \mathbf{C}_2; \mathbf{r}(0) = \mathbf{i} - \mathbf{j} + 2\mathbf{k} = \mathbf{C}_2$$

$$\Rightarrow \mathbf{r}(t) = \left(t^2 + \frac{4}{\sqrt{6}}t + 1\right)\mathbf{i} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t - 1\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t + 2\right)\mathbf{k} = \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)(2\mathbf{i} + \mathbf{j} + \mathbf{k}) + (\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

31. $\mathbf{v} = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{a} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; $|\mathbf{v}|^2 = (1 - \cos t)^2 + \sin^2 t = 2 - 2\cos t \Rightarrow |\mathbf{v}|^2$ is at a max when $\cos t = -1 \Rightarrow t = \pi, 3\pi, 5\pi$, etc., and at these values of t , $|\mathbf{v}|^2 = 4 \Rightarrow \max |\mathbf{v}| = \sqrt{4} = 2$; $|\mathbf{v}|^2$ is at a min when $\cos t = 1 \Rightarrow t = 0, 2\pi, 4\pi$, etc., and at these values of t , $|\mathbf{v}|^2 = 0 \Rightarrow \min |\mathbf{v}| = 0$; $|\mathbf{a}|^2 = \sin^2 t + \cos^2 t = 1$ for every $t \Rightarrow \max |\mathbf{a}| = \min |\mathbf{a}| = \sqrt{1} = 1$

32. Let $\mathbf{p} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ denote the position vector of the point, $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ and $\mathbf{v} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$.

Then $\mathbf{r}(t) = \mathbf{p} + (\cos t)\mathbf{u} + (\sin t)\mathbf{v}$. Note that $(2, 2, 1)$ is a point on the plane and $\mathbf{n} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is normal to the plane. Moreover, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors with $\mathbf{u} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n} = 0 \Rightarrow \mathbf{u}$ and $\mathbf{v}$ are parallel to the plane. Therefore, $\mathbf{r}(t)$ identifies a point that lies in the plane for each t . Also, for each t , $(\cos t)\mathbf{u} + (\sin t)\mathbf{v}$ is a unit vector. Starting at the point $(2, 2, 1)$ the vector $(\cos t)\mathbf{u} + (\sin t)\mathbf{v}$ traces out a circle of radius 1 and center $(2, 2, 1)$ in the plane $x + y - 2z = 2$.

33. $\mathbf{v} = (-3\sin t)\mathbf{j} + (2\cos t)\mathbf{k}$ and $\mathbf{a} = (-3\cos t)\mathbf{j} - (2\sin t)\mathbf{k}$; $|\mathbf{v}|^2 = 9\sin^2 t + 4\cos^2 t \Rightarrow \frac{d}{dt}(|\mathbf{v}|^2)$
 $= 18\sin t \cos t - 8\cos t \sin t = 10\sin t \cos t$; $\frac{d}{dt}(|\mathbf{v}|^2) = 0 \Rightarrow 10\sin t \cos t = 0 \Rightarrow \sin t = 0$ or $\cos t = 0$
 $\Rightarrow t = 0, \pi$ or $t = \frac{\pi}{2}, \frac{3\pi}{2}$. When $t = 0, \pi$, $|\mathbf{v}|^2 = 4 \Rightarrow |\mathbf{v}| = \sqrt{4} = 2$; when $t = \frac{\pi}{2}, \frac{3\pi}{2}$, $|\mathbf{v}|^2 = 9 \Rightarrow |\mathbf{v}| = \sqrt{9} = 3$.
Therefore $\max |\mathbf{v}|$ is 3 when $t = \frac{\pi}{2}, \frac{3\pi}{2}$, and $\min |\mathbf{v}| = 2$ when $t = 0, \pi$. Next, $|\mathbf{a}|^2 = 9\cos^2 t + 4\sin^2 t$
 $\Rightarrow \frac{d}{dt}(|\mathbf{a}|^2) = -18\cos t \sin t + 8\sin t \cos t = -10\sin t \cos t$; $\frac{d}{dt}(|\mathbf{a}|^2) = 0 \Rightarrow -10\sin t \cos t = 0 \Rightarrow \sin t = 0$ or
 $\cos t = 0 \Rightarrow t = 0, \pi$ or $t = \frac{\pi}{2}, \frac{3\pi}{2}$. When $t = 0, \pi$, $|\mathbf{a}|^2 = 9 \Rightarrow |\mathbf{a}| = 3$; when $t = \frac{\pi}{2}, \frac{3\pi}{2}$, $|\mathbf{a}|^2 = 4 \Rightarrow |\mathbf{a}| = 2$.
Therefore, $\max |\mathbf{a}| = 3$ when $t = 0, \pi$, and $\min |\mathbf{a}| = 2$ when $t = \frac{\pi}{2}, \frac{3\pi}{2}$.

34. $\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = 2\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = 2 \cdot 0 = 0 \Rightarrow \mathbf{v} \cdot \mathbf{v}$ is a constant $\Rightarrow |\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ is constant

35. $\mathbf{u} = C\mathbf{a} + b\mathbf{j} + c\mathbf{k}$ with a, b, c real constants $\Rightarrow \frac{d\mathbf{u}}{dt} = \frac{da}{dt}\mathbf{i} + \frac{db}{dt}\mathbf{j} + \frac{dc}{dt}\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

$$36. \text{ (a) } \mathbf{u} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \Rightarrow c\mathbf{u} = cf(t)\mathbf{i} + cg(t)\mathbf{j} + ch(t)\mathbf{k} \Rightarrow \frac{d}{dt}(c\mathbf{u}) = c \frac{df}{dt}\mathbf{i} + c \frac{dg}{dt}\mathbf{j} + c \frac{dh}{dt}\mathbf{k} \\ = c \left(\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k} \right) = c \frac{d\mathbf{u}}{dt}$$

$$\text{ (b) } f\mathbf{u} = f[f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}] \Rightarrow \frac{d}{dt}(f\mathbf{u}) = \left[\frac{df}{dt}f(t) + f \frac{df}{dt} \right] \mathbf{i} + \left[\frac{df}{dt}g(t) + f \frac{dg}{dt} \right] \mathbf{j} + \left[\frac{df}{dt}h(t) + f \frac{dh}{dt} \right] \mathbf{k} \\ = \frac{df}{dt}[f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}] + f \left[\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k} \right] = \frac{df}{dt}\mathbf{u} + f \frac{d\mathbf{u}}{dt}$$

37. Let $\mathbf{u} = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ and $\mathbf{v} = g_1(t)\mathbf{i} + g_2(t)\mathbf{j} + g_3(t)\mathbf{k}$. Then

$$\mathbf{u} + \mathbf{v} = [f_1(t) + g_1(t)]\mathbf{i} + [f_2(t) + g_2(t)]\mathbf{j} + [f_3(t) + g_3(t)]\mathbf{k} \\ \Rightarrow \frac{d}{dt}(\mathbf{u} + \mathbf{v}) = [f_1'(t) + g_1'(t)]\mathbf{i} + [f_2'(t) + g_2'(t)]\mathbf{j} + [f_3'(t) + g_3'(t)]\mathbf{k} \\ = [f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k}] + [g_1'(t)\mathbf{i} + g_2'(t)\mathbf{j} + g_3'(t)\mathbf{k}] = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}; \\ \mathbf{u} - \mathbf{v} = [f_1(t) - g_1(t)]\mathbf{i} + [f_2(t) - g_2(t)]\mathbf{j} + [f_3(t) - g_3(t)]\mathbf{k} \\ \Rightarrow \frac{d}{dt}(\mathbf{u} - \mathbf{v}) = [f_1'(t) - g_1'(t)]\mathbf{i} + [f_2'(t) - g_2'(t)]\mathbf{j} + [f_3'(t) - g_3'(t)]\mathbf{k} \\ = [f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k}] - [g_1'(t)\mathbf{i} + g_2'(t)\mathbf{j} + g_3'(t)\mathbf{k}] = \frac{d\mathbf{u}}{dt} - \frac{d\mathbf{v}}{dt}$$

38. Suppose $\mathbf{r}$ is continuous at $t = t_0$. Then $\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{r}(t_0) \Leftrightarrow \lim_{t \rightarrow t_0} [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]$

$$= f(t_0)\mathbf{i} + g(t_0)\mathbf{j} + h(t_0)\mathbf{k} \Leftrightarrow \lim_{t \rightarrow t_0} f(t) = f(t_0), \lim_{t \rightarrow t_0} g(t) = g(t_0), \text{ and } \lim_{t \rightarrow t_0} h(t) = h(t_0) \Leftrightarrow f, g, \text{ and } h \text{ are} \\ \text{continuous at } t = t_0.$$

$$39. \lim_{t \rightarrow t_0} [\mathbf{r}_1(t) \times \mathbf{r}_2(t)] = \lim_{t \rightarrow t_0} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1(t) & f_2(t) & f_3(t) \\ g_1(t) & g_2(t) & g_3(t) \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \lim_{t \rightarrow t_0} f_1(t) & \lim_{t \rightarrow t_0} f_2(t) & \lim_{t \rightarrow t_0} f_3(t) \\ \lim_{t \rightarrow t_0} g_1(t) & \lim_{t \rightarrow t_0} g_2(t) & \lim_{t \rightarrow t_0} g_3(t) \end{vmatrix} \\ = \lim_{t \rightarrow t_0} \mathbf{r}_1(t) \times \lim_{t \rightarrow t_0} \mathbf{r}_2(t) = \mathbf{u} \times \mathbf{v}$$

40. $\mathbf{r}'(t_0)$ exists $\Rightarrow f'(t_0)\mathbf{i} + g'(t_0)\mathbf{j} + h'(t_0)\mathbf{k}$ exists $\Rightarrow f'(t_0), g'(t_0), h'(t_0)$ all exist $\Rightarrow f, g,$ and h are continuous at $t = t_0 \Rightarrow \mathbf{r}(t)$ is continuous at $t = t_0$

$$41. \text{ (a) } \frac{d}{dt}(\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}) = \frac{d\mathbf{u}}{dt} \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot \frac{d}{dt}(\mathbf{v} \times \mathbf{w}) = \frac{d\mathbf{u}}{dt} \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot \left(\frac{d\mathbf{v}}{dt} \times \mathbf{w} + \mathbf{v} \times \frac{d\mathbf{w}}{dt} \right) \\ = \frac{d\mathbf{u}}{dt} \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot \frac{d\mathbf{v}}{dt} \times \mathbf{w} + \mathbf{u} \cdot \mathbf{v} \times \frac{d\mathbf{w}}{dt}$$

(b) Each of the determinants is equivalent to each expression in Eq. (7) in part (a) because of the determinant formula for the Triple Scalar Product in Section 10.2.

$$42. \frac{d}{dt} \left[\mathbf{r} \cdot \left(\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} \right) \right] = \frac{d\mathbf{r}}{dt} \cdot \left(\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} \right) + \mathbf{r} \cdot \left(\frac{d^2\mathbf{r}}{dt^2} \times \frac{d^2\mathbf{r}}{dt^2} \right) + \mathbf{r} \cdot \left(\frac{d\mathbf{r}}{dt} \times \frac{d^3\mathbf{r}}{dt^3} \right) = \mathbf{r} \cdot \left(\frac{d\mathbf{r}}{dt} \times \frac{d^3\mathbf{r}}{dt^3} \right), \text{ since } \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$$

and $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{v}) = 0$ for any vectors $\mathbf{u}$ and $\mathbf{v}$

$$\begin{aligned}
 43. \quad (a) \quad \int_a^b \mathbf{k} \mathbf{r}(t) \, dt &= \int_a^b [\mathbf{k}f(t)\mathbf{i} + \mathbf{k}g(t)\mathbf{j} + \mathbf{k}h(t)\mathbf{k}] \, dt = \int_a^b [\mathbf{k}f(t)] \, dt \mathbf{i} + \int_a^b [\mathbf{k}g(t)] \, dt \mathbf{j} + \int_a^b [\mathbf{k}h(t)] \, dt \mathbf{k} \\
 &= \mathbf{k} \left(\int_a^b f(t) \, dt \mathbf{i} + \int_a^b g(t) \, dt \mathbf{j} + \int_a^b h(t) \, dt \mathbf{k} \right) = \mathbf{k} \int_a^b \mathbf{r}(t) \, dt
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \int_a^b [\mathbf{r}_1(t) \pm \mathbf{r}_2(t)] \, dt &= \int_a^b ([f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k}] \pm [f_2(t)\mathbf{i} + g_2(t)\mathbf{j} + h_2(t)\mathbf{k}]) \, dt \\
 &= \int_a^b ([f_1(t) \pm f_2(t)]\mathbf{i} + [g_1(t) \pm g_2(t)]\mathbf{j} + [h_1(t) \pm h_2(t)]\mathbf{k}) \, dt \\
 &= \int_a^b [f_1(t) \pm f_2(t)] \, dt \mathbf{i} + \int_a^b [g_1(t) \pm g_2(t)] \, dt \mathbf{j} + \int_a^b [h_1(t) \pm h_2(t)] \, dt \mathbf{k} \\
 &= \left[\int_a^b f_1(t) \, dt \mathbf{i} \pm \int_a^b f_2(t) \, dt \mathbf{i} \right] + \left[\int_a^b g_1(t) \, dt \mathbf{j} \pm \int_a^b g_2(t) \, dt \mathbf{j} \right] + \left[\int_a^b h_1(t) \, dt \mathbf{k} \pm \int_a^b h_2(t) \, dt \mathbf{k} \right] \\
 &= \int_a^b \mathbf{r}_1(t) \, dt \pm \int_a^b \mathbf{r}_2(t) \, dt
 \end{aligned}$$

$$\begin{aligned}
 (c) \quad \text{Let } \mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}. \text{ Then } \int_a^b \mathbf{C} \cdot \mathbf{r}(t) \, dt &= \int_a^b [c_1f(t) + c_2g(t) + c_3h(t)] \, dt \\
 &= c_1 \int_a^b f(t) \, dt + c_2 \int_a^b g(t) \, dt + c_3 \int_a^b h(t) \, dt = \mathbf{C} \cdot \int_a^b \mathbf{r}(t) \, dt; \\
 \int_a^b \mathbf{C} \times \mathbf{r}(t) \, dt &= \int_a^b [c_2h(t) - c_3g(t)]\mathbf{i} + [c_3f(t) - c_1h(t)]\mathbf{j} + [c_1g(t) - c_2f(t)]\mathbf{k} \, dt \\
 &= \left[c_2 \int_a^b h(t) \, dt - c_3 \int_a^b g(t) \, dt \right] \mathbf{i} + \left[c_3 \int_a^b f(t) \, dt - c_1 \int_a^b h(t) \, dt \right] \mathbf{j} + \left[c_1 \int_a^b g(t) \, dt - c_2 \int_a^b f(t) \, dt \right] \mathbf{k} \\
 &= \mathbf{C} \times \int_a^b \mathbf{r}(t) \, dt
 \end{aligned}$$

$$\begin{aligned}
 44. \quad (a) \quad \text{Let } u \text{ and } \mathbf{r} \text{ be continuous on } [a, b]. \text{ Then } \lim_{t \rightarrow t_0} u(t)\mathbf{r}(t) &= \lim_{t \rightarrow t_0} [u(t)f(t)\mathbf{i} + u(t)g(t)\mathbf{j} + u(t)h(t)\mathbf{k}] \\
 &= u(t_0)f(t_0)\mathbf{i} + u(t_0)g(t_0)\mathbf{j} + u(t_0)h(t_0)\mathbf{k} = u(t_0)\mathbf{r}(t_0) \Rightarrow u\mathbf{r} \text{ is continuous for every } t_0 \text{ in } [a, b].
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \text{Let } u \text{ and } \mathbf{r} \text{ be differentiable. Then } \frac{d}{dt}(u\mathbf{r}) &= \frac{d}{dt}[u(t)f(t)\mathbf{i} + u(t)g(t)\mathbf{j} + u(t)h(t)\mathbf{k}] \\
 &= \left(\frac{du}{dt}f(t) + u(t)\frac{df}{dt} \right) \mathbf{i} + \left(\frac{du}{dt}g(t) + u(t)\frac{dg}{dt} \right) \mathbf{j} + \left(\frac{du}{dt}h(t) + u(t)\frac{dh}{dt} \right) \mathbf{k}
 \end{aligned}$$

$$= [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}] \frac{d\mathbf{u}}{dt} + \mathbf{u}(t) \left(\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k} \right) = \mathbf{r} \frac{d\mathbf{u}}{dt} + \mathbf{u} \frac{d\mathbf{r}}{dt}$$

45. (a) If $\mathbf{R}_1(t)$ and $\mathbf{R}_2(t)$ have identical derivatives on $\mathbf{i}$, then $\frac{d\mathbf{R}_1}{dt} = \frac{df_1}{dt}\mathbf{i} + \frac{dg_1}{dt}\mathbf{j} + \frac{dh_1}{dt}\mathbf{k} = \frac{df_2}{dt}\mathbf{i} + \frac{dg_2}{dt}\mathbf{j} + \frac{dh_2}{dt}\mathbf{k}$
 $= \frac{d\mathbf{R}_2}{dt} \Rightarrow \frac{df_1}{dt} = \frac{df_2}{dt}, \frac{dg_1}{dt} = \frac{dg_2}{dt}, \frac{dh_1}{dt} = \frac{dh_2}{dt} \Rightarrow f_1(t) = f_2(t) + c_1, g_1(t) = g_2(t) + c_2, h_1(t) = h_2(t) + c_3$
 $\Rightarrow f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k} = [f_2(t) + c_1]\mathbf{i} + [g_2(t) + c_2]\mathbf{j} + [h_2(t) + c_3]\mathbf{k} \Rightarrow \mathbf{R}_1(t) = \mathbf{R}_2(t) + \mathbf{C}$, where
 $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

- (b) Let $\mathbf{R}(t)$ be an antiderivative of $\mathbf{r}(t)$ on $\mathbf{i}$. Then $\mathbf{R}'(t) = \mathbf{r}(t)$. If $\mathbf{U}(t)$ is an antiderivative of $\mathbf{r}(t)$ on $\mathbf{i}$, then $\mathbf{U}'(t) = \mathbf{r}(t)$. Thus $\mathbf{U}'(t) = \mathbf{R}'(t)$ on $\mathbf{i} \Rightarrow \mathbf{U}(t) = \mathbf{R}(t) + \mathbf{C}$.

$$46. \frac{d}{dt} \int_a^t \mathbf{r}(\tau) d\tau = \frac{d}{dt} \int_a^t [f(\tau)\mathbf{i} + g(\tau)\mathbf{j} + h(\tau)\mathbf{k}] d\tau = \frac{d}{dt} \int_a^t f(\tau) d\tau \mathbf{i} + \frac{d}{dt} \int_a^t g(\tau) d\tau \mathbf{j} + \frac{d}{dt} \int_a^t h(\tau) d\tau \mathbf{k}$$

$$= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} = \mathbf{r}(t). \text{ Since } \frac{d}{dt} \int_a^t \mathbf{r}(\tau) d\tau = \mathbf{r}(t), \text{ we have that } \int_a^t \mathbf{r}(\tau) d\tau \text{ is an antiderivative of}$$

$$\mathbf{r}. \text{ If } \mathbf{R} \text{ is any antiderivative of } \mathbf{r}, \text{ then } \mathbf{R}(t) = \int_a^t \mathbf{r}(\tau) d\tau + \mathbf{C} \text{ by Exercise 45(b). Then } \mathbf{R}(a) = \int_a^a \mathbf{r}(\tau) d\tau + \mathbf{C}$$

$$= \mathbf{0} + \mathbf{C} \Rightarrow \mathbf{C} = \mathbf{R}(a) \Rightarrow \int_a^t \mathbf{r}(\tau) d\tau = \mathbf{R}(t) - \mathbf{C} = \mathbf{R}(t) - \mathbf{R}(a) \Rightarrow \int_a^b \mathbf{r}(\tau) d\tau = \mathbf{R}(b) - \mathbf{R}(a).$$

47-50. Example CAS commands:

Maple:

```
with(plots):
x:= t -> sin(t) - t*cos(t);
y:= t -> cos(t) + t*sin(t);
z:= t -> t^2;
s1:= spacecurve([x(t),y(t),z(t)], t=0..6*Pi, numpoints = 120, axes=NORMAL);
dx:= t -> D(x)(t);
dy:= t -> D(y)(t);
dz:= t -> D(z)(t);
t0:= 3*Pi/2;
s2:=spacecurve([x(t0)+t*dx(t0),y(t0)+t*dy(t0),z(t0)+t*dz(t0),t=-2..2]);
display({s1,s2},title = 'Space Curve and Tangent Line at t0=3 Pi/2');
```

Mathematica:

```
Clear[x,y,z,t]
r[t_] = {x[t],y[t],z[t]}
x[t_] = Sin[t] - t Cos[t]
y[t_] = Cos[t] + t Sin[t]
z[t_] = t^2
{a,b} = {0, 6 Pi};
t0 = 3/2 Pi;
p1 = ParametricPlot3D[ {x[t],y[t],z[t]}, {t,a,b} ]
v[t_] = r'[t]
v0 = v[t0]
```

```

line[t_] = r[t0] + t v0
p2 = ParametricPlot3D[ Evaluate[ line[t] ], {t,-2,2} ]
Show[ p1, p2 ]

```

51-52. Example CAS commands:

Maple:

```

with(plots):
x:= t -> cos(a*t):
y:= t -> sin(a*t):
z:= t -> b*t: a:=2: b:= 1:
s1:=spacecurve([x(t),y(t),z(t)], t=0..4*Pi, numpoints = 400, axes=NORMAL):
dx:= t -> D(x)(t);
dy:= t -> D(y)(t);
dz:= t -> D(z)(t);
t0:= 3*Pi/2:
s2:=spacecurve([x(t0)+t*dx(t0),y(t0)+t*dy(t0),z(t0)+t*dz(t0)],t=-2..2):
display({s1,s2},title = `Helix With a = 2 and b = 1`);

```

Mathematica:

```

Clear[a,b]
x[t_] = Cos[a t]
y[t_] = Sin[a t]
z[t_] = b t
t0 = 3/2 Pi;
v[t_] = r'[t]
v0 = v[t0]
line[t_] = r[t0] + t v0
b = 1
a = 2
p1 = ParametricPlot3D[ {x[t],y[t],z[t]}, {t,0,4Pi} ]
p2 = ParametricPlot3D[ Evaluate[ line[t] ], {t,-2,2} ]
Show[ p1, p2 ]

```

10.6 ARC LENGTH AND THE UNIT TANGENT VECTOR T

$$\begin{aligned}
 1. \quad \mathbf{r} &= (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + \sqrt{5}t\mathbf{k} \Rightarrow \mathbf{v} = (-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + \sqrt{5}\mathbf{k} \\
 \Rightarrow |\mathbf{v}| &= \sqrt{(-2 \sin t)^2 + (2 \cos t)^2 + (\sqrt{5})^2} = \sqrt{4 \sin^2 t + 4 \cos^2 t + 5} = 3; \quad \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} \\
 &= \left(-\frac{2}{3} \sin t\right)\mathbf{i} + \left(\frac{2}{3} \cos t\right)\mathbf{j} + \frac{\sqrt{5}}{3}\mathbf{k} \text{ and Length} = \int_0^\pi |\mathbf{v}| \, dt = \int_0^\pi 3 \, dt = [3t]_0^\pi = 3\pi
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \mathbf{r} &= (6 \sin 2t)\mathbf{i} + (6 \cos 2t)\mathbf{j} + 5t\mathbf{k} \Rightarrow \mathbf{v} = (12 \cos 2t)\mathbf{i} + (-12 \sin 2t)\mathbf{j} + 5\mathbf{k} \\
 \Rightarrow |\mathbf{v}| &= \sqrt{(12 \cos 2t)^2 + (-12 \sin 2t)^2 + 5^2} = \sqrt{144 \cos^2 2t + 144 \sin^2 2t + 25} = 13; \quad \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} \\
 &= \left(\frac{12}{13} \cos 2t\right)\mathbf{i} - \left(\frac{12}{13} \sin 2t\right)\mathbf{j} + \frac{5}{13}\mathbf{k} \text{ and Length} = \int_0^\pi |\mathbf{v}| \, dt = \int_0^\pi 13 \, dt = [13t]_0^\pi = 13\pi
 \end{aligned}$$

$$3. \mathbf{r} = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + t^{1/2}\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (t^{1/2})^2} = \sqrt{1+t}; \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1+t}}\mathbf{i} + \frac{\sqrt{t}}{\sqrt{1+t}}\mathbf{k}$$

$$\text{and Length} = \int_0^8 \sqrt{1+t} \, dt = \left[\frac{2}{3}(1+t)^{3/2} \right]_0^8 = \frac{52}{3}$$

$$4. \mathbf{r} = (2+t)\mathbf{i} - (t+1)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}; \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

$$\text{and Length} = \int_0^3 \sqrt{3} \, dt = [\sqrt{3}t]_0^3 = 3\sqrt{3}$$

$$5. \mathbf{r} = (\cos^3 t)\mathbf{j} + (\sin^3 t)\mathbf{k} \Rightarrow \mathbf{v} = (-3\cos^2 t \sin t)\mathbf{j} + (3\sin^2 t \cos t)\mathbf{k} \Rightarrow |\mathbf{v}|$$

$$= \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2} = \sqrt{(9\cos^2 t \sin^2 t)(\cos^2 t + \sin^2 t)} = 3|\cos t \sin t|;$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-3\cos^2 t \sin t}{3|\cos t \sin t|}\mathbf{j} + \frac{3\sin^2 t \cos t}{3|\cos t \sin t|}\mathbf{k} = (-\cos t)\mathbf{j} + (\sin t)\mathbf{k}, \text{ if } 0 \leq t \leq \frac{\pi}{2}, \text{ and}$$

$$\text{Length} = \int_0^{\pi/2} 3|\cos t \sin t| \, dt = \int_0^{\pi/2} 3\cos t \sin t \, dt = \int_0^{\pi/2} \frac{3}{2} \sin 2t \, dt = \left[-\frac{3}{4} \cos 2t \right]_0^{\pi/2} = \frac{3}{2}$$

$$6. \mathbf{r} = 6t^3\mathbf{i} - 2t^3\mathbf{j} - 3t^3\mathbf{k} \Rightarrow \mathbf{v} = 18t^2\mathbf{i} - 6t^2\mathbf{j} - 9t^2\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(18t^2)^2 + (-6t^2)^2 + (-9t^2)^2} = \sqrt{441t^4} = 21t^2;$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{18t^2}{21t^2}\mathbf{i} - \frac{6t^2}{21t^2}\mathbf{j} - \frac{9t^2}{21t^2}\mathbf{k} = \frac{6}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} - \frac{3}{7}\mathbf{k} \text{ and Length} = \int_1^2 21t^2 \, dt = [7t^3]_1^2 = 49$$

$$7. \mathbf{r} = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} + \frac{2\sqrt{2}}{3}t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = (\cos t - t \sin t)\mathbf{i} + (\sin t + t \cos t)\mathbf{j} + (\sqrt{2}t^{1/2})\mathbf{k}$$

$$\Rightarrow |\mathbf{v}| = \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + (\sqrt{2}t)^2} = \sqrt{1 + t^2 + 2t} = \sqrt{(t+1)^2} = |t+1| = t+1, \text{ if } t \geq 0;$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\cos t - t \sin t}{t+1} \right)\mathbf{i} + \left(\frac{\sin t + t \cos t}{t+1} \right)\mathbf{j} + \left(\frac{\sqrt{2}t^{1/2}}{t+1} \right)\mathbf{k} \text{ and Length} = \int_0^\pi (t+1) \, dt = \left[\frac{t^2}{2} + t \right]_0^\pi = \frac{\pi^2}{2} + \pi$$

$$8. \mathbf{r} = (t \sin t + \cos t)\mathbf{i} + (t \cos t - \sin t)\mathbf{j} \Rightarrow \mathbf{v} = (\sin t + t \cos t - \sin t)\mathbf{i} + (\cos t - t \sin t - \cos t)\mathbf{j}$$

$$= (t \cos t)\mathbf{i} - (t \sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t \cos t)^2 + (-t \sin t)^2} = \sqrt{t^2} = |t| = t \text{ if } \sqrt{2} \leq t \leq 2; \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

$$= \left(\frac{t \cos t}{t} \right)\mathbf{i} - \left(\frac{t \sin t}{t} \right)\mathbf{j} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} \text{ and Length} = \int_{\sqrt{2}}^2 t \, dt = \left[\frac{t^2}{2} \right]_{\sqrt{2}}^2 = 1$$

$$9. \text{ Let } P(t_0) \text{ denote the point. Then } \mathbf{v} = (5 \cos t)\mathbf{i} - (5 \sin t)\mathbf{j} + 12\mathbf{k} \text{ and } 26\pi = \int_0^{t_0} \sqrt{25 \cos^2 t + 25 \sin^2 t + 144} \, dt$$

$$= \int_0^{t_0} 13 \, dt = 13t_0 \Rightarrow t_0 = 2\pi, \text{ and the point is } P(2\pi) = (5 \sin 2\pi, 5 \cos 2\pi, 24\pi) = (5, 0, 24\pi)$$

10. Let $P(t_0)$ denote the point. Then $\mathbf{v} = (12 \cos t)\mathbf{i} + (12 \sin t)\mathbf{j} + 5\mathbf{k}$ and

$$-13\pi = \int_0^{t_0} \sqrt{144 \cos^2 t + 144 \sin^2 t + 25} \, dt = \int_0^{t_0} 13 \, dt = 13t_0 \Rightarrow t_0 = -\pi, \text{ and the point is}$$

$$P(-\pi) = (12 \sin(-\pi), -12 \cos(-\pi), -5\pi) = (0, 12, -5\pi)$$

11. $\mathbf{r} = (4 \cos t)\mathbf{i} + (4 \sin t)\mathbf{j} + 3t\mathbf{k} \Rightarrow \mathbf{v} = (-4 \sin t)\mathbf{i} + (4 \cos t)\mathbf{j} + 3\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-4 \sin t)^2 + (4 \cos t)^2 + 3^2}$

$$= \sqrt{25} = 5 \Rightarrow s(t) = \int_0^t 5 \, d\tau = 5t \Rightarrow \text{Length} = s\left(\frac{\pi}{2}\right) = \frac{5\pi}{2}$$

12. $\mathbf{r} = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j} \Rightarrow \mathbf{v} = (-\sin t + \sin t + t \cos t)\mathbf{i} + (\cos t - \cos t + t \sin t)\mathbf{j}$

$$= (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t \cos t)^2 + (t \sin t)^2} = \sqrt{t^2} = t, \text{ since } \frac{\pi}{2} \leq t \leq \pi \Rightarrow s(t) = \int_0^t \tau \, d\tau = \frac{t^2}{2}$$

$$\Rightarrow \text{Length} = s(\pi) - s\left(\frac{\pi}{2}\right) = \frac{\pi^2}{2} - \frac{\left(\frac{\pi}{2}\right)^2}{2} = \frac{3\pi^2}{8}$$

13. $\mathbf{r} = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + e^t\mathbf{k} \Rightarrow \mathbf{v} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} + e^t\mathbf{k}$

$$\Rightarrow |\mathbf{v}| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + (e^t)^2} = \sqrt{3e^{2t}} = \sqrt{3}e^t \Rightarrow s(t) = \int_0^t \sqrt{3}e^{\tau} \, d\tau$$

$$= \sqrt{3}e^t - \sqrt{3} \Rightarrow \text{Length} = s(0) - s(-\ln 4) = 0 - (\sqrt{3}e^{-\ln 4} - \sqrt{3}) = \frac{3\sqrt{3}}{4}$$

14. $\mathbf{r} = (1 + 2t)\mathbf{i} + (1 + 3t)\mathbf{j} + (6 - 6t)\mathbf{k} \Rightarrow \mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{2^2 + 3^2 + (-6)^2} = 7 \Rightarrow s(t) = \int_0^t 7 \, d\tau = 7t$

$$\Rightarrow \text{Length} = s(0) - s(-1) = 0 - (-7) = 7$$

15. $\mathbf{r} = t\mathbf{i} + \ln(\cos t)\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + \left(\frac{-\sin t}{\cos t}\right)\mathbf{j} = \mathbf{i} - (\tan t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (-\tan t)^2} = \sqrt{\sec^2 t} = |\sec t| = \sec t$, since

$$-\frac{\pi}{2} < t < \frac{\pi}{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sec t}\right)\mathbf{i} - \left(\frac{\tan t}{\sec t}\right)\mathbf{j} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}; \quad \frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}$$

$$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j};$$

$$\Rightarrow \kappa = \frac{1}{|\mathbf{v}|} \left|\frac{d\mathbf{T}}{dt}\right| = \left(\frac{1}{\sec t}\right) \cdot 1 = \cos t$$

$$16. \mathbf{r} = \ln(\sec t)\mathbf{i} + t\mathbf{j} \Rightarrow \mathbf{v} = \left(\frac{\sec t \tan t}{\sec t}\right)\mathbf{i} + \mathbf{j} = (\tan t)\mathbf{i} + \mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(\tan t)^2 + 1^2} = \sqrt{\sec^2 t} = |\sec t| = \sec t,$$

$$\text{since } -\frac{\pi}{2} < t < \frac{\pi}{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\tan t}{\sec t}\right)\mathbf{i} - \left(\frac{1}{\sec t}\right)\mathbf{j} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}; \frac{d\mathbf{T}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$$

$$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(\cos t)^2 + (-\sin t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j};$$

$$\Rightarrow \kappa = \frac{1}{|\mathbf{v}|} \left|\frac{d\mathbf{T}}{dt}\right| = \left(\frac{1}{\sec t}\right) \cdot 1 = \cos t$$

$$17. \mathbf{r} = (2t+3)\mathbf{i} + (5-t^2)\mathbf{j} \Rightarrow \mathbf{v} = 2\mathbf{i} - 2t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{2^2 + (-2t)^2} = 2\sqrt{1+t^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{2\sqrt{1+t^2}}\mathbf{i} + \frac{-2t}{2\sqrt{1+t^2}}\mathbf{j}$$

$$= \frac{1}{\sqrt{1+t^2}}\mathbf{i} - \frac{t}{\sqrt{1+t^2}}\mathbf{j}; \frac{d\mathbf{T}}{dt} = \frac{-t}{(\sqrt{1+t^2})^3} - \frac{1}{(\sqrt{1+t^2})^3}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{-t}{(\sqrt{1+t^2})^3}\right)^2 + \left(\frac{-1}{(\sqrt{1+t^2})^3}\right)^2}$$

$$= \sqrt{\frac{1}{(1+t^2)^2}} = \frac{1}{1+t^2} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{-t}{\sqrt{1+t^2}}\mathbf{i} - \frac{1}{\sqrt{1+t^2}}\mathbf{j};$$

$$\Rightarrow \kappa = \frac{1}{|\mathbf{v}|} \left|\frac{d\mathbf{T}}{dt}\right| = \left(\frac{1}{2\sqrt{1+t^2}}\right) \left(\frac{1}{1+t^2}\right) = \frac{1}{2(1+t^2)^{3/2}}$$

$$18. \mathbf{r} = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j} \Rightarrow \mathbf{v} = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t \cos t)^2 + (t \sin t)^2} = \sqrt{t^2} = |t|$$

$$= t, \text{ since } t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(t \cos t)\mathbf{i} + (t \sin t)\mathbf{j}}{t} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}; \frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$$

$$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j};$$

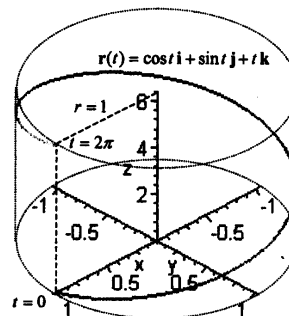
$$\Rightarrow \kappa = \frac{1}{|\mathbf{v}|} \left|\frac{d\mathbf{T}}{dt}\right| = \left(\frac{1}{t}\right) \cdot 1 = \frac{1}{t}$$

$$= [(t \cos t)(\sin t + t \cos t) - (t \sin t)(\cos t - t \sin t)]\mathbf{k} = t^2\mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}| = \sqrt{(t^2)^2} = t^2 \Rightarrow \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{t^2}{t^3} = \frac{1}{t}$$

$$19. \mathbf{r} = (\sqrt{2}t)\mathbf{i} + (\sqrt{2}t)\mathbf{j} + (1-t^2)\mathbf{k} \Rightarrow \mathbf{v} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} - 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2 + (-2t)^2} = \sqrt{4+4t^2}$$

$$= 2\sqrt{1+t^2} \Rightarrow \text{Length} = \int_0^1 2\sqrt{1+t^2} dt = \left[2\left(\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln(t + \sqrt{1+t^2})\right)\right]_0^1 = \sqrt{2} + \ln(1 + \sqrt{2})$$

20. Let the helix make one complete turn from $t = 0$ to $t = 2\pi$. Note that the radius of the cylinder is 1 $\Rightarrow$ the circumference of the base is 2π . When $t = 2\pi$, the point P is $(\cos 2\pi, \sin 2\pi, 2\pi) = (1, 0, 2\pi) \Rightarrow$ the cylinder is 2π units high. Cut the cylinder along PQ and flatten. The resulting rectangle has a width equal to the circumference of the cylinder $= 2\pi$ and a height equal to 2π , the height of the cylinder. Therefore, the rectangle is a square and the portion of the helix from $t = 0$ to $t = 2\pi$ is its diagonal.



21. (a) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (1 - \cos t)\mathbf{k}$, $0 \leq t \leq 2\pi \Rightarrow x = \cos t$, $y = \sin t$, $z = 1 - \cos t \Rightarrow x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, a right circular cylinder with the z -axis as the axis and radius = 1. Therefore $P(\cos t, \sin t, 1 - \cos t)$ lies on the cylinder $x^2 + y^2 = 1$; $t = 0 \Rightarrow P(1, 0, 0)$ is on the curve; $t = \frac{\pi}{2} \Rightarrow Q(0, 1, 1)$ is on the curve; $t = \pi \Rightarrow R(-1, 0, 2)$ is on the curve. Then $\vec{PQ} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\vec{PR} = -2\mathbf{i} + 2\mathbf{k}$

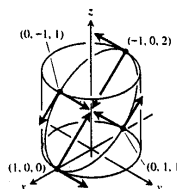
$$\Rightarrow \vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ -2 & 0 & 2 \end{vmatrix} = 2\mathbf{i} + 2\mathbf{k} \text{ is a vector normal to the plane of } P, Q, \text{ and } R. \text{ Then the}$$

plane containing P, Q, and R has an equation $2x + 2z = 2(1) + 2(0)$ or $x + z = 1$. Any point on the curve will satisfy this equation since $x + z = \cos t + (1 - \cos t) = 1$. Therefore, any point on the curve lies on the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + z = 1 \Rightarrow$ the curve is an ellipse.

- (b) $\mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\sin t)\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\sin^2 t + \cos^2 t + \sin^2 t} = \sqrt{1 + \sin^2 t} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

$$= \frac{(-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\sin t)\mathbf{k}}{\sqrt{1 + \sin^2 t}} \Rightarrow \mathbf{T}(0) = \mathbf{j}, \mathbf{T}\left(\frac{\pi}{2}\right) = \frac{-\mathbf{i} + \mathbf{k}}{\sqrt{2}}, \mathbf{T}(\pi) = -\mathbf{j}, \mathbf{T}\left(\frac{3\pi}{2}\right) = \frac{\mathbf{i} - \mathbf{k}}{\sqrt{2}}$$

- (c) $\mathbf{a} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} + (\cos t)\mathbf{k}$; $\mathbf{n} = \mathbf{i} + \mathbf{k}$ is normal to the plane $x + z = 1 \Rightarrow \mathbf{n} \cdot \mathbf{a} = -\cos t + \cos t = 0 \Rightarrow \mathbf{a}$ is orthogonal to $\mathbf{n} \Rightarrow \mathbf{a}$ is parallel to the plane; $\mathbf{a}(0) = -\mathbf{i} + \mathbf{k}$, $\mathbf{a}\left(\frac{\pi}{2}\right) = -\mathbf{j}$, $\mathbf{a}\left(\frac{3\pi}{2}\right) = \mathbf{j}$



- (d) $|\mathbf{v}| = \sqrt{1 + \sin^2 t}$ (See part (b)) $\Rightarrow L = \int_0^{2\pi} \sqrt{1 + \sin^2 t} dt$

- (e) $L \approx 7.64$ (by *Mathematica*)