

CHAPTER 7 TECHNIQUES OF INTEGRATION, L'HÔPITAL'S RULE, AND IMPROPER INTEGRALS

7.1 BASIC INTEGRATION FORMULAS

$$1. \int \frac{16x \, dx}{\sqrt{8x^2 + 1}}; \left[\begin{array}{l} u = 8x^2 + 1 \\ du = 16x \, dx \end{array} \right] \rightarrow \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{8x^2 + 1} + C$$

$$2. \int \frac{3 \cos x \, dx}{\sqrt{1 + 3 \sin x}}; \left[\begin{array}{l} u = 1 + 3 \sin x \\ du = 3 \cos x \, dx \end{array} \right] \rightarrow \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{1 + 3 \sin x} + C$$

$$3. \int 3\sqrt{\sin v} \cos v \, dv; \left[\begin{array}{l} u = \sin v \\ du = \cos v \, dv \end{array} \right] \rightarrow \int 3\sqrt{u} \, du = 3 \cdot \frac{2}{3} u^{3/2} + C = 2(\sin v)^{3/2} + C$$

$$4. \int \cot^3 y \csc^2 y \, dy; \left[\begin{array}{l} u = \cot y \\ du = -\csc^2 y \, dy \end{array} \right] \rightarrow \int u^3 (-du) = -\frac{u^4}{4} + C = \frac{-\cot^4 y}{4} + C$$

$$5. \int_0^1 \frac{16x \, dx}{8x^2 + 2}; \left[\begin{array}{l} u = 8x^2 + 2 \\ du = 16x \, dx \\ x = 0 \Rightarrow u = 2, \quad x = 1 \Rightarrow u = 10 \end{array} \right] \rightarrow \int_2^{10} \frac{du}{u} = [\ln |u|]_2^{10} = \ln 10 - \ln 2 = \ln 5$$

$$6. \int_{\pi/4}^{\pi/3} \frac{\sec^2 z \, dz}{\tan z}; \left[\begin{array}{l} u = \tan z \\ du = \sec^2 z \, dz \\ z = \pi/4 \Rightarrow u = 1, \quad z = \pi/3 \Rightarrow u = \sqrt{3} \end{array} \right] \rightarrow \int_1^{\sqrt{3}} \frac{1}{u} \, du = [\ln |u|]_1^{\sqrt{3}} = \ln \sqrt{3} - \ln 1 = \ln \sqrt{3}$$

$$7. \int \frac{dx}{\sqrt{x}(\sqrt{x} + 1)}; \left[\begin{array}{l} u = \sqrt{x} \\ du = \frac{1}{2\sqrt{x}} \, dx \\ 2 \, du = \frac{dx}{\sqrt{x}} \end{array} \right] \rightarrow \int \frac{2 \, du}{u} = 2 \ln |u| + C = 2 \ln(\sqrt{x} + 1) + C$$

$$8. \int \frac{dx}{x - \sqrt{x}} = \int \frac{dx}{\sqrt{x}(\sqrt{x} - 1)}; \left[\begin{array}{l} u = \sqrt{x} - 1 \\ du = \frac{1}{2\sqrt{x}} \, dx \\ 2 \, du = \frac{dx}{\sqrt{x}} \end{array} \right] \rightarrow \int \frac{2 \, du}{u} = 2 \ln |u| + C = 2 \ln |\sqrt{x} - 1| + C$$

$$9. \int \cot(3-7x) dx; \left[\begin{array}{l} u = 3-7x \\ du = -7 dx \end{array} \right] \rightarrow -\frac{1}{7} \int \cot u du = -\frac{1}{7} \ln |\sin u| + C = -\frac{1}{7} \ln |\sin(3-7x)| + C$$

$$10. \int \csc(\pi x - 1) dx; \left[\begin{array}{l} u = \pi x - 1 \\ du = \pi dx \end{array} \right] \rightarrow \int \csc u \cdot \frac{du}{\pi} = \frac{-1}{\pi} \ln |\csc u + \cot u| + C \\ = -\frac{1}{\pi} \ln |\csc(\pi x - 1) + \cot(\pi x - 1)| + C$$

$$11. \int e^\theta \csc(e^\theta + 1) d\theta; \left[\begin{array}{l} u = e^\theta + 1 \\ du = e^\theta d\theta \end{array} \right] \rightarrow \int \csc u du = -\ln |\csc u + \cot u| + C = -\ln |\csc(e^\theta + 1) + \cot(e^\theta + 1)| + C$$

$$12. \int \frac{\cot(3 + \ln x)}{x} dx; \left[\begin{array}{l} u = 3 + \ln x \\ du = \frac{dx}{x} \end{array} \right] \rightarrow \int \cot u du = \ln |\sin u| + C = \ln |\sin(3 + \ln x)| + C$$

$$13. \int \sec \frac{t}{3} dt; \left[\begin{array}{l} u = \frac{t}{3} \\ du = \frac{dt}{3} \end{array} \right] \rightarrow \int 3 \sec u du = 3 \ln |\sec u + \tan u| + C = 3 \ln \left| \sec \frac{t}{3} + \tan \frac{t}{3} \right| + C$$

$$14. \int x \sec(x^2 - 5) dx; \left[\begin{array}{l} u = x^2 - 5 \\ du = 2x dx \end{array} \right] \rightarrow \int \frac{1}{2} \sec u du = \frac{1}{2} \ln |\sec u + \tan u| + C \\ = \frac{1}{2} \ln |\sec(x^2 - 5) + \tan(x^2 - 5)| + C$$

$$15. \int \csc(s - \pi) ds; \left[\begin{array}{l} u = s - \pi \\ du = ds \end{array} \right] \rightarrow \int \csc u du = -\ln |\csc u + \cot u| + C = -\ln |\csc(s - \pi) + \cot(s - \pi)| + C$$

$$16. \int \frac{1}{\theta^2} \csc \frac{1}{\theta} d\theta; \left[\begin{array}{l} u = \frac{1}{\theta} \\ du = -\frac{d\theta}{\theta^2} \end{array} \right] \rightarrow \int -\csc u du = \ln |\csc u + \cot u| + C = \ln \left| \csc \frac{1}{\theta} + \cot \frac{1}{\theta} \right| + C$$

$$17. \int_0^{\sqrt{\ln 2}} 2xe^{x^2} dx; \left[\begin{array}{l} u = x^2 \\ du = 2x dx \\ x = 0 \Rightarrow u = 0, x = \sqrt{\ln 2} \Rightarrow u = \ln 2 \end{array} \right] \rightarrow \int_0^{\ln 2} e^u du = [e^u]_0^{\ln 2} = e^{\ln 2} - e^0 = 2 - 1 = 1$$

$$18. \int_{\pi/2}^{\pi} \sin(y) e^{\cos y} dy; \left[\begin{array}{l} u = \cos y \\ du = -\sin y dy \\ x = \frac{\pi}{2} \Rightarrow u = 0, x = \pi \Rightarrow u = -1 \end{array} \right] \rightarrow \int_0^{-1} -e^u du = \int_{-1}^0 e^u du = [e^u]_{-1}^0 = 1 - e^{-1} = \frac{e-1}{e}$$

$$19. \int e^{\tan v} \sec^2 v dv; \left[\begin{array}{l} u = \tan v \\ du = \sec^2 v dv \end{array} \right] \rightarrow \int e^u du = e^u + C = e^{\tan v} + C$$

$$20. \int \frac{e^{\sqrt{t}} dt}{\sqrt{t}}; \left[\begin{array}{l} u = \sqrt{t} \\ du = \frac{dt}{2\sqrt{t}} \end{array} \right] \rightarrow \int 2e^u du = 2e^u + C = 2e^{\sqrt{t}} + C$$

$$21. \int 3^{x+1} dx; \left[\begin{array}{l} u = x+1 \\ du = dx \end{array} \right] \rightarrow \int 3^u du = \left(\frac{1}{\ln 3} \right) 3^u + C = \frac{3^{(x+1)}}{\ln 3} + C$$

$$22. \int \frac{2^{\ln x}}{x} dx; \left[\begin{array}{l} u = \ln x \\ du = \frac{dx}{x} \end{array} \right] \rightarrow \int 2^u du = \frac{2^u}{\ln 2} + C = \frac{2^{\ln x}}{\ln 2} + C$$

$$23. \int \frac{2^{\sqrt{w}} dw}{2\sqrt{w}}; \left[\begin{array}{l} u = \sqrt{w} \\ du = \frac{dw}{2\sqrt{w}} \end{array} \right] \rightarrow \int 2^u du = \frac{2^u}{\ln 2} + C = \frac{2^{\sqrt{w}}}{\ln 2} + C$$

$$24. \int 10^{2\theta} d\theta; \left[\begin{array}{l} u = 2\theta \\ du = 2 d\theta \end{array} \right] \rightarrow \int \frac{1}{2} 10^u du = \frac{10^u}{2 \ln 10} + C = \frac{1}{2} \left(\frac{10^{2\theta}}{\ln 10} \right) + C$$

$$25. \int \frac{9 du}{1+9u^2}; \left[\begin{array}{l} x = 3u \\ dx = 3 du \end{array} \right] \rightarrow \int \frac{3 dx}{1+x^2} = 3 \tan^{-1} x + C = 3 \tan^{-1} 3u + C$$

$$26. \int \frac{4 dx}{1+(2x+1)^2}; \left[\begin{array}{l} u = 2x+1 \\ du = 2 dx \end{array} \right] \rightarrow \int \frac{2 du}{1+u^2} = 2 \tan^{-1} u + C = 2 \tan^{-1} (2x+1) + C$$

$$27. \int_0^{1/6} \frac{dx}{\sqrt{1-9x^2}}; \left[\begin{array}{l} u = 3x \\ du = 3 dx \\ x=0 \Rightarrow u=0, x=\frac{1}{6} \Rightarrow u=\frac{1}{2} \end{array} \right] \rightarrow \int_0^{1/2} \frac{1}{3} \frac{du}{\sqrt{1-u^2}} = \left[\frac{1}{3} \sin^{-1} u \right]_0^{1/2} = \frac{1}{3} \left(\frac{\pi}{6} - 0 \right) = \frac{\pi}{18}$$

$$28. \int_0^1 \frac{dt}{\sqrt{4-t^2}} = \left[\sin^{-1} \frac{t}{2} \right]_0^1 = \sin^{-1} \left(\frac{1}{2} \right) - 0 = \frac{\pi}{6}$$

$$29. \int \frac{2s ds}{\sqrt{1-s^4}}; \left[\begin{array}{l} u = s^2 \\ du = 2s ds \end{array} \right] \rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1} s^2 + C$$

$$30. \int \frac{2 dx}{x\sqrt{1-4 \ln^2 x}}; \left[\begin{array}{l} u = 2 \ln x \\ du = \frac{2 dx}{x} \end{array} \right] \rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1} (2 \ln x) + C$$

$$31. \int \frac{6 \, dx}{x\sqrt{25x^2-1}} = \int \frac{6 \, dx}{5x\sqrt{x^2-\frac{1}{25}}} = \frac{6}{5} \cdot 5 \sec^{-1} |5x| + C = 6 \sec^{-1} |5x| + C$$

$$32. \int \frac{dr}{r\sqrt{r^2-9}} = \frac{1}{3} \sec^{-1} \left| \frac{r}{3} \right| + C$$

$$33. \int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x dx}{e^{2x} + 1}; \left[\begin{array}{l} u = e^x \\ du = e^x dx \end{array} \right] \rightarrow \int \frac{du}{u^2 + 1} = \tan^{-1} u + C = \tan^{-1} e^x + C$$

$$34. \int \frac{dy}{\sqrt{e^{2y}-1}} = \int \frac{e^y dy}{e^y \sqrt{(e^y)^2-1}}; \left[\begin{array}{l} u = e^y \\ du = e^y dy \end{array} \right] \rightarrow \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} |u| + C = \sec^{-1} e^y + C$$

$$35. \int_1^{e^{\pi/3}} \frac{dx}{x \cos(\ln x)}; \left[\begin{array}{l} u = \ln x \\ du = \frac{dx}{x} \\ x = 1 \Rightarrow u = 0, x = e^{\pi/3} \Rightarrow u = \frac{\pi}{3} \end{array} \right] \rightarrow \int_0^{\pi/3} \frac{du}{\cos u} = \int_0^{\pi/3} \sec u \, du = [\ln |\sec u + \tan u|]_0^{\pi/3}$$

$$= \ln \left| \sec \frac{\pi}{3} + \tan \frac{\pi}{3} \right| - \ln |\sec 0 + \tan 0| = \ln(2 + \sqrt{3}) - \ln(1) = \ln(2 + \sqrt{3})$$

$$36. \int \frac{\ln x \, dx}{x + 4x \ln^2 x} = \int \frac{\ln x \, dx}{x(1 + 4 \ln^2 x)}; \left[\begin{array}{l} u = \ln^2 x \\ du = \frac{2}{x} \ln x \, dx \end{array} \right] \rightarrow \int \frac{1}{2} \frac{du}{1 + 4u} = \frac{1}{8} \ln |1 + 4u| + C = \frac{1}{8} \ln(1 + 4 \ln^2 x) + C$$

$$37. \int_1^2 \frac{8 \, dx}{x^2 - 2x + 2} = 8 \int_1^2 \frac{dx}{1 + (x-1)^2}; \left[\begin{array}{l} u = x-1 \\ du = dx \\ x = 1 \Rightarrow u = 0, x = 2 \Rightarrow u = 1 \end{array} \right] \rightarrow 8 \int_0^1 \frac{du}{1 + u^2} = 8 [\tan^{-1} u]_0^1$$

$$= 8(\tan^{-1} 1 - \tan^{-1} 0) = 8\left(\frac{\pi}{4} - 0\right) = 2\pi$$

$$38. \int_2^4 \frac{2 \, dx}{x^2 - 6x + 10} = 2 \int_2^4 \frac{dx}{(x-3)^2 + 1}; \left[\begin{array}{l} u = x-3 \\ du = dx \\ x = 2 \Rightarrow u = -1, x = 4 \Rightarrow u = 1 \end{array} \right] \rightarrow 2 \int_{-1}^1 \frac{du}{u^2 + 1} = 2 [\tan^{-1} u]_{-1}^1$$

$$= 2[\tan^{-1} 1 - \tan^{-1}(-1)] = 2\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \pi$$

$$39. \int \frac{dt}{\sqrt{-t^2 + 4t - 3}} = \int \frac{dt}{\sqrt{1 - (t-2)^2}}; \left[\begin{array}{l} u = t-2 \\ du = dt \end{array} \right] \rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(t-2) + C$$

$$40. \int \frac{d\theta}{\sqrt{2\theta - \theta^2}} = \int \frac{d\theta}{\sqrt{1 - (\theta-1)^2}}; \left[\begin{array}{l} u = \theta-1 \\ du = d\theta \end{array} \right] \rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(\theta-1) + C$$

$$41. \int \frac{dx}{(x+1)\sqrt{x^2+2x}} = \int \frac{dx}{(x+1)\sqrt{(x+1)^2-1}}; \left[\begin{array}{l} u = x+1 \\ du = dx \end{array} \right] \rightarrow \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C = \sec^{-1}|x+1| + C,$$

$$|u| = |x+1| > 1$$

$$42. \int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} = \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}}; \left[\begin{array}{l} u = x-2 \\ du = dx \end{array} \right] \rightarrow \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C$$

$$= \sec^{-1}|x+2| + C, |u| = |x+2| > 1$$

$$43. \int (\sec x + \cot x)^2 dx = \int (\sec^2 x + 2 \sec x \cot x + \cot^2 x) dx = \int \sec^2 x dx + \int 2 \csc x dx + \int (\csc^2 x - 1) dx$$

$$= \tan x - 2 \ln |\csc x + \cot x| - \cot x - x + C$$

$$44. \int (\csc x - \tan x)^2 dx = \int (\csc^2 x - 2 \csc x \tan x + \tan^2 x) dx = \int \csc^2 x dx - \int 2 \sec x dx + \int (\sec^2 x - 1) dx$$

$$= -\cot x - 2 \ln |\sec x + \tan x| + \tan x - x + C$$

$$45. \int \csc x \sin 3x dx = \int (\csc x)(\sin 2x \cos x + \sin x \cos 2x) dx = \int (\csc x)(2 \sin x \cos^2 x + \sin x \cos 2x) dx$$

$$= \int (2 \cos^2 x + \cos 2x) dx = \int [(1 + \cos 2x) + \cos 2x] dx = \int (1 + 2 \cos 2x) dx = x + \sin 2x + C$$

$$46. \int (\sin 3x \cos 2x - \cos 3x \sin 2x) dx = \int \sin(3x - 2x) dx = \int \sin x dx = -\cos x + C$$

$$47. \int \frac{x}{x+1} dx = \int \left(1 - \frac{1}{x+1}\right) dx = x - \ln|x+1| + C$$

$$48. \int \frac{x^2}{x^2+1} dx = \int \left(1 - \frac{1}{x^2+1}\right) dx = x - \tan^{-1} x + C$$

$$49. \int_{\sqrt{2}}^3 \frac{2x^3}{x^2-1} dx = \int_{\sqrt{2}}^3 \left(2x + \frac{2x}{x^2-1}\right) dx = [x^2 + \ln|x^2-1|]_{\sqrt{2}}^3 = (9 + \ln 8) - (2 + \ln 1) = 7 + \ln 8$$

$$50. \int_{-1}^3 \frac{4x^2-7}{2x+3} dx = \int_{-1}^3 \left[(2x-3) + \frac{2}{2x+3}\right] dx = [x^2 - 3x + \ln|2x+3|]_{-1}^3 = (9 - 9 + \ln 9) - (1 + 3 + \ln 1) = \ln 9 - 4$$

$$51. \int \frac{4t^3 - t^2 + 16t}{t^2 + 4} dt = \int \left[(4t-1) + \frac{4}{t^2+4}\right] dt = 2t^2 - t + 2 \tan^{-1}\left(\frac{t}{2}\right) + C$$

$$52. \int \frac{2\theta^3 - 7\theta^2 + 7\theta}{2\theta - 5} d\theta = \int \left[(\theta^2 - \theta + 1) + \frac{5}{2\theta - 5}\right] d\theta = \frac{\theta^3}{3} - \frac{\theta^2}{2} + \theta + \frac{5}{2} \ln|2\theta - 5| + C$$

$$53. \int \frac{1-x}{\sqrt{1-x^2}} dx = \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x dx}{\sqrt{1-x^2}} = \sin^{-1} x + \sqrt{1-x^2} + C$$

$$54. \int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}} dx = \int \frac{dx}{2\sqrt{x-1}} + \int \frac{dx}{x} = (x-1)^{1/2} + \ln|x| + C$$

$$55. \int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x} dx = \int_0^{\pi/4} (\sec^2 x + \sec x \tan x) dx = [\tan x + \sec x]_0^{\pi/4} = (1+\sqrt{2}) - (0+1) = \sqrt{2}$$

$$56. \int_0^{1/2} \frac{2-8x}{1+4x^2} dx = \int_0^{1/2} \left(\frac{2}{1+4x^2} - \frac{8x}{1+4x^2} \right) dx = \left[\tan^{-1}(2x) - \ln|1+4x^2| \right]_0^{1/2} \\ = (\tan^{-1} 1 - \ln 2) - (\tan^{-1} 0 - \ln 1) = \frac{\pi}{4} - \ln 2$$

$$57. \int \frac{dx}{1+\sin x} = \int \frac{(1-\sin x)}{(1-\sin^2 x)} dx = \int \frac{(1-\sin x)}{\cos^2 x} dx = \int (\sec^2 x - \sec x \tan x) dx = \tan x - \sec x + C$$

$$58. 1 + \cos x = 1 + \cos\left(2 \cdot \frac{x}{2}\right) = 2 \cos^2 \frac{x}{2} \Rightarrow \int \frac{dx}{1+\cos x} = \int \frac{dx}{2 \cos^2(\frac{x}{2})} = \frac{1}{2} \int \sec^2\left(\frac{x}{2}\right) dx = \tan \frac{x}{2} + C$$

$$59. \int \frac{1}{\sec \theta + \tan \theta} d\theta = \int \frac{\cos \theta}{1 + \sin \theta} d\theta; \left[\begin{array}{l} u = 1 + \sin \theta \\ du = \cos \theta d\theta \end{array} \right] \rightarrow \int \frac{du}{u} = \ln|u| + C = \ln|1 + \sin \theta| + C$$

$$60. \int \frac{1}{\csc \theta + \cot \theta} d\theta = \int \frac{\sin \theta}{1 + \cos \theta} d\theta; \left[\begin{array}{l} u = 1 + \cos \theta \\ du = -\sin \theta d\theta \end{array} \right] \rightarrow \int \frac{-du}{u} = -\ln|u| + C = -\ln|1 + \cos \theta| + C$$

$$61. \int \frac{1}{1-\sec x} dx = \int \frac{\cos x}{\cos x - 1} dx = \int \left(1 + \frac{1}{\cos x - 1} \right) dx = \int \left(1 - \frac{1 + \cos x}{\sin^2 x} \right) dx = \int \left(1 - \csc^2 x - \frac{\cos x}{\sin^2 x} \right) dx \\ = \int (1 - \csc^2 x - \csc x \cot x) dx = x + \cot x + \csc x + C$$

$$62. \int \frac{1}{1-\csc x} dx = \int \frac{\sin x}{\sin x - 1} dx = \int \left(1 + \frac{1}{\sin x - 1} \right) dx = \int \left(1 + \frac{\sin x + 1}{(\sin x - 1)(\sin x + 1)} \right) dx \\ = \int \left(1 - \frac{1 + \sin x}{\cos^2 x} \right) dx = \int \left(1 - \sec^2 x - \frac{\sin x}{\cos^2 x} \right) dx = \int (1 - \sec^2 x - \sec x \tan x) dx = x - \tan x - \sec x + C$$

$$63. \int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}} dx = \int_0^{2\pi} \left| \sin \frac{x}{2} \right| dx; \left[\begin{array}{l} \sin \frac{x}{2} \geq 0 \\ \text{for } 0 \leq \frac{x}{2} \leq \pi \end{array} \right] \rightarrow \int_0^{2\pi} \sin\left(\frac{x}{2}\right) dx = \left[-2 \cos \frac{x}{2} \right]_0^{2\pi} = -2(\cos \pi - \cos 0) \\ = (-2)(-2) = 4$$

$$64. \int_0^{\pi} \sqrt{1 - \cos 2x} \, dx = \int_0^{\pi} \sqrt{2} |\sin x| \, dx; \left[\begin{array}{l} \sin x \geq 0 \\ \text{for } 0 \leq x \leq \pi \end{array} \right] \rightarrow \sqrt{2} \int_0^{\pi} \sin x \, dx = [-\sqrt{2} \cos x]_0^{\pi} \\ = -\sqrt{2}(\cos \pi - \cos 0) = 2\sqrt{2}$$

$$65. \int_{\pi/2}^{\pi} \sqrt{1 + \cos 2t} \, dt = \int_{\pi/2}^{\pi} \sqrt{2} |\cos t| \, dt; \left[\begin{array}{l} \cos t \leq 0 \\ \text{for } \frac{\pi}{2} \leq t \leq \pi \end{array} \right] \rightarrow \int_{\pi/2}^{\pi} -\sqrt{2} \cos t \, dt = [-\sqrt{2} \sin t]_{\pi/2}^{\pi} \\ = -\sqrt{2}(\sin \pi - \sin \frac{\pi}{2}) = \sqrt{2}$$

$$66. \int_{-\pi}^0 \sqrt{1 + \cos t} \, dt = \int_{-\pi}^0 \sqrt{2} \left| \cos \frac{t}{2} \right| \, dt; \left[\begin{array}{l} \cot \frac{t}{2} \geq 0 \\ \text{for } -\pi \leq t \leq 0 \end{array} \right] \rightarrow \int_{-\pi}^0 \sqrt{2} \cos \frac{t}{2} \, dt = [2\sqrt{2} \sin \frac{t}{2}]_{-\pi}^0 \\ = 2\sqrt{2}[\sin 0 - \sin(-\frac{\pi}{2})] = 2\sqrt{2}$$

$$67. \int_{-\pi}^0 \sqrt{1 - \cos^2 \theta} \, d\theta = \int_{-\pi}^0 |\sin \theta| \, d\theta; \left[\begin{array}{l} \sin \theta \leq 0 \\ \text{for } -\pi \leq \theta \leq 0 \end{array} \right] \rightarrow \int_{-\pi}^0 -\sin \theta \, d\theta = [\cos \theta]_{-\pi}^0 = \cos 0 - \cos(-\pi) \\ = 1 - (-1) = 2$$

$$68. \int_{\pi/2}^{\pi} \sqrt{1 - \sin^2 \theta} \, d\theta = \int_{\pi/2}^{\pi} |\cos \theta| \, d\theta; \left[\begin{array}{l} \cos \theta \leq 0 \\ \text{for } \frac{\pi}{2} \leq \theta \leq \pi \end{array} \right] \rightarrow \int_{\pi/2}^{\pi} -\cos \theta \, d\theta = [-\sin \theta]_{\pi/2}^{\pi} = -\sin \pi + \sin \frac{\pi}{2} = 1$$

$$69. \int_{-\pi/4}^{\pi/4} \sqrt{\tan^2 y + 1} \, dy = \int_{-\pi/4}^{\pi/4} |\sec y| \, dy; \left[\begin{array}{l} \sec y \geq 0 \\ \text{for } -\frac{\pi}{4} \leq y \leq \frac{\pi}{4} \end{array} \right] \rightarrow \int_{-\pi/4}^{\pi/4} \sec y \, dy = [\ln |\sec y + \tan y|]_{-\pi/4}^{\pi/4} \\ = \ln |\sqrt{2} + 1| - \ln |\sqrt{2} - 1|$$

$$70. \int_{-\pi/4}^0 \sqrt{\sec^2 y - 1} \, dy = \int_{-\pi/4}^0 |\tan y| \, dy; \left[\begin{array}{l} \tan y \leq 0 \\ \text{for } -\frac{\pi}{4} \leq y \leq 0 \end{array} \right] \rightarrow \int_{-\pi/4}^0 -\tan y \, dy = [\ln |\cos y|]_{-\pi/4}^0 = -\ln\left(\frac{1}{\sqrt{2}}\right) \\ = \ln \sqrt{2}$$

$$71. \int_{\pi/4}^{3\pi/4} (\csc x - \cot x)^2 \, dx = \int_{\pi/4}^{3\pi/4} (\csc^2 x - 2 \csc x \cot x + \cot^2 x) \, dx = \int_{\pi/4}^{3\pi/4} (2 \csc^2 x - 1 - 2 \csc x \cot x) \, dx \\ = [-2 \cot x - x + 2 \csc x]_{\pi/4}^{3\pi/4} = \left(-2 \cot \frac{3\pi}{4} - \frac{3\pi}{4} + 2 \csc \frac{3\pi}{4}\right) - \left(-2 \cot \frac{\pi}{4} - \frac{\pi}{4} + 2 \csc \frac{\pi}{4}\right) \\ = [-2(-1) - \frac{3\pi}{4} + 2(\sqrt{2})] - [-2(1) - \frac{\pi}{4} + 2(\sqrt{2})] = 4 - \frac{\pi}{2}$$

$$\begin{aligned}
 72. \int_0^{\pi/4} (\sec x + 4 \cos x)^2 dx &= \int_0^{\pi/4} \left[\sec^2 x + 8 + 16 \left(\frac{1 + \sin 2x}{2} \right) \right] dx = [\tan x + 16x - 4 \cos 2x]_0^{\pi/4} \\
 &= \left(\tan \frac{\pi}{4} + 4\pi - 4 \cos \frac{\pi}{2} \right) - (\tan 0 + 0 - 4 \cos 0) = 5 + 4\pi
 \end{aligned}$$

$$\begin{aligned}
 73. \int \cos \theta \csc(\sin \theta) d\theta; \left[\begin{array}{l} u = \sin \theta \\ du = \cos \theta d\theta \end{array} \right] &\rightarrow \int \csc u du = -\ln |\csc u + \cot u| + C \\
 &= -\ln |\csc(\sin \theta) + \cot(\sin \theta)| + C
 \end{aligned}$$

$$74. \int \left(1 + \frac{1}{x} \right) \cot(x + \ln x) dx; \left[\begin{array}{l} u = x + \ln x \\ du = \left(1 + \frac{1}{x} \right) dx \end{array} \right] \rightarrow \int \cot u du = \ln |\sin u| + C = \ln |\sin(x + \ln x)| + C$$

$$\begin{aligned}
 75. \int (\csc x - \sec x)(\sin x + \cos x) dx &= \int (1 + \cot x - \tan x - 1) dx = \int \cot x dx - \int \tan x dx \\
 &= \ln |\sin x| + \ln |\cos x| + C
 \end{aligned}$$

$$76. \int 3 \sinh \left(\frac{x}{2} + \ln 5 \right) dx = \left[\begin{array}{l} u = \frac{x}{2} + \ln 5 \\ 2 du = dx \end{array} \right] = 6 \int \sinh u du = 6 \cosh u + C = 6 \cosh \left(\frac{x}{2} + \ln 5 \right) + C$$

$$77. \int \frac{6 dy}{\sqrt{y}(1+y)}; \left[\begin{array}{l} u = \sqrt{y} \\ du = \frac{1}{2\sqrt{y}} dy \end{array} \right] \rightarrow \int \frac{12 du}{1+u^2} = 12 \tan^{-1} u + C = 12 \tan^{-1} \sqrt{y} + C$$

$$78. \int \frac{dx}{x\sqrt{4x^2-1}} = \int \frac{2 dx}{2x\sqrt{(2x)^2-1}}; \left[\begin{array}{l} u = 2x \\ du = 2 dx \end{array} \right] \rightarrow \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} |u| + C = \sec^{-1} |2x| + C$$

$$\begin{aligned}
 79. \int \frac{7 dx}{(x-1)\sqrt{x^2-2x-48}} &= \int \frac{7 dx}{(x-1)\sqrt{(x-1)^2-49}}; \left[\begin{array}{l} u = x-1 \\ du = dx \end{array} \right] \rightarrow \int \frac{7 du}{u\sqrt{u^2-49}} = 7 \cdot \frac{1}{7} \sec^{-1} \left| \frac{u}{7} \right| + C \\
 &= \sec^{-1} \left| \frac{x-1}{7} \right| + C
 \end{aligned}$$

$$\begin{aligned}
 80. \int \frac{dx}{(2x+1)\sqrt{4x^2+4x}} &= \int \frac{dx}{(2x+1)\sqrt{(2x+1)^2-1}}; \left[\begin{array}{l} u = 2x+1 \\ du = 2 dx \end{array} \right] \rightarrow \int \frac{du}{2u\sqrt{u^2-1}} = \frac{1}{2} \sec^{-1} |u| + C \\
 &= \frac{1}{2} \sec^{-1} |2x+1| + C
 \end{aligned}$$

$$81. \int \sec^2 t \tan(\tan t) dt; \left[\begin{array}{l} u = \tan t \\ du = \sec^2 t dt \end{array} \right] \rightarrow \int \tan u du = -\ln |\cos u| + C = \ln |\sec u| + C = \ln |\sec(\tan t)| + C$$

$$82. \int \frac{dx}{x\sqrt{3+x^2}} = -\frac{1}{\sqrt{3}} \operatorname{csch}^{-1} \left| \frac{x}{\sqrt{3}} \right| + C \text{ (from Table 6.15)}$$

$$83. (a) \int \cos^3 \theta \, d\theta = \int (\cos \theta)(1 - \sin^2 \theta) \, d\theta; \left[\begin{array}{l} u = \sin \theta \\ du = \cos \theta \, d\theta \end{array} \right] \rightarrow \int (1 - u^2) \, du = u - \frac{u^3}{3} + C = \sin \theta - \frac{1}{3} \sin^3 \theta + C$$

$$(b) \int \cos^5 \theta \, d\theta = \int (\cos \theta)(1 - \sin^2 \theta)^2 \, d\theta = \int (1 - u^2)^2 \, du = \int (1 - 2u^2 + u^4) \, du = u - \frac{2}{3}u^3 + \frac{u^5}{5} + C \\ = \sin \theta - \frac{2}{3} \sin^3 \theta + \frac{1}{5} \sin^5 \theta + C$$

$$(c) \int \cos^9 \theta \, d\theta = \int (\cos^8 \theta)(\cos \theta) \, d\theta = \int (1 - \sin^2 \theta)^4 (\cos \theta) \, d\theta$$

$$84. (a) \int \sin^3 \theta \, d\theta = \int (1 - \cos^2 \theta)(\sin \theta) \, d\theta; \left[\begin{array}{l} u = \cos \theta \\ du = -\sin \theta \, d\theta \end{array} \right] \rightarrow \int (1 - u^2)(-du) = \frac{u^3}{3} - u + C \\ = -\cos \theta + \frac{1}{3} \cos^3 \theta + C$$

$$(b) \int \sin^5 \theta \, d\theta = \int (1 - \cos^2 \theta)^2 (\sin \theta) \, d\theta = \int (1 - u^2)^2 (-du) = \int (-1 + 2u^2 - u^4) \, du \\ = -\cos \theta + \frac{2}{3} \cos^3 \theta - \frac{1}{5} \cos^5 \theta + C$$

$$(c) \int \sin^7 \theta \, d\theta = \int (1 - u^2)^3 (-du) = \int (-1 + 3u^2 - 3u^4 + u^6) \, du = -\cos \theta + \cos^3 \theta - \frac{3}{5} \cos^5 \theta + \frac{\cos^7 \theta}{7} + C$$

$$(d) \int \sin^{13} \theta \, d\theta = \int (\sin^{12} \theta)(\sin \theta) \, d\theta = \int (1 - \cos^2 \theta)^6 (\sin \theta) \, d\theta$$

$$85. (a) \int \tan^3 \theta \, d\theta = \int (\sec^2 \theta - 1)(\tan \theta) \, d\theta = \int \sec^2 \theta \tan \theta \, d\theta - \int \tan \theta \, d\theta = \frac{1}{2} \tan^2 \theta - \int \tan \theta \, d\theta \\ = \frac{1}{2} \tan^2 \theta + \ln |\cos \theta| + C$$

$$(b) \int \tan^5 \theta \, d\theta = \int (\sec^2 \theta - 1)(\tan^3 \theta) \, d\theta = \int \tan^3 \theta \sec^2 \theta \, d\theta - \int \tan^3 \theta \, d\theta = \frac{1}{4} \tan^4 \theta - \int \tan^3 \theta \, d\theta$$

$$(c) \int \tan^7 \theta \, d\theta = \int (\sec^2 \theta - 1)(\tan^5 \theta) \, d\theta = \int \tan^5 \theta \sec^2 \theta \, d\theta - \int \tan^5 \theta \, d\theta = \frac{1}{6} \tan^6 \theta - \int \tan^5 \theta \, d\theta$$

$$(d) \int \tan^{2k+1} \theta \, d\theta = \int (\sec^2 \theta - 1)(\tan^{2k-1} \theta) \, d\theta = \int \tan^{2k-1} \theta \sec^2 \theta \, d\theta - \int \tan^{2k-1} \theta \, d\theta;$$

$$\left[\begin{array}{l} u = \tan \theta \\ du = \sec^2 \theta \, d\theta \end{array} \right] \rightarrow \int u^{2k-1} \, du - \int \tan^{2k-1} \theta \, d\theta = \frac{1}{2k} u^{2k} - \int \tan^{2k-1} \theta \, d\theta = \frac{1}{2k} \tan^{2k} \theta - \int \tan^{2k-1} \theta \, d\theta$$

$$86. (a) \int \cot^3 \theta \, d\theta = \int (\csc^2 \theta - 1)(\cot \theta) \, d\theta = \int \cot \theta \csc^2 \theta \, d\theta - \int \cot \theta \, d\theta = -\frac{1}{2} \cot^2 \theta - \int \cot \theta \, d\theta \\ = -\frac{1}{2} \cot^2 \theta - \ln |\sin \theta| + C$$

$$(b) \int \cot^5 \theta \, d\theta = \int (\csc^2 \theta - 1)(\cot^3 \theta) \, d\theta = \int \cot^3 \theta \csc^2 \theta \, d\theta - \int \cot^3 \theta \, d\theta = -\frac{1}{4} \cot^4 \theta - \int \cot^3 \theta \, d\theta$$

$$(c) \int \cot^7 \theta \, d\theta = \int (\csc^2 \theta - 1)(\cot^5 \theta) \, d\theta = \int \cot^5 \theta \csc^2 \theta \, d\theta - \int \cot^5 \theta \, d\theta = -\frac{1}{6} \cot^6 \theta - \int \cot^5 \theta \, d\theta$$

$$(d) \int \cot^{2k+1} \theta \, d\theta = \int (\csc^2 \theta - 1)(\cot^{2k-1} \theta) \, d\theta = \int \cot^{2k-1} \theta \csc^2 \theta \, d\theta - \int \cot^{2k-1} \theta \, d\theta;$$

$$\left[\begin{array}{l} u = \cot \theta \\ du = -\csc^2 \theta \, d\theta \end{array} \right] \rightarrow - \int u^{2k-1} \, du - \int \cot^{2k-1} \theta \, d\theta = -\frac{1}{2k} u^{2k} - \int \cot^{2k-1} \theta \, d\theta$$

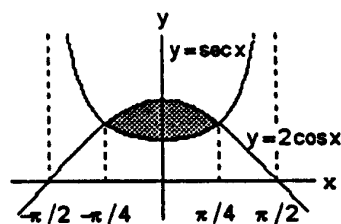
$$= -\frac{1}{2k} \cot^{2k} \theta - \int \cot^{2k-1} \theta \, d\theta$$

$$87. A = \int_{-\pi/4}^{\pi/4} (2 \cos x - \sec x) \, dx = [2 \sin x - \ln |\sec x + \tan x|]_{-\pi/4}^{\pi/4}$$

$$= [\sqrt{2} - \ln(\sqrt{2} + 1)] - [-\sqrt{2} - \ln(\sqrt{2} - 1)]$$

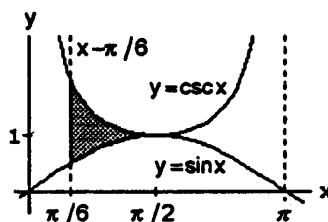
$$= 2\sqrt{2} - \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = 2\sqrt{2} - \ln\left(\frac{(\sqrt{2}+1)^2}{2-1}\right)$$

$$= 2\sqrt{2} - \ln(3 + 2\sqrt{2})$$



$$88. A = \int_{\pi/6}^{\pi/2} (\csc x - \sin x) \, dx = [-\ln |\csc x + \cot x| + \cos x]_{\pi/6}^{\pi/2}$$

$$= -\ln |1 + 0| + \ln |2 + \sqrt{3}| - \frac{\sqrt{3}}{2} = \ln(2 + \sqrt{3}) - \frac{\sqrt{3}}{2}$$



$$89. V = \int_{-\pi/4}^{\pi/4} \pi(2 \cos x)^2 \, dx - \int_{-\pi/4}^{\pi/4} \pi \sec^2 x \, dx = 4\pi \int_{-\pi/4}^{\pi/4} \cos^2 x \, dx - \pi \int_{-\pi/4}^{\pi/4} \sec^2 x \, dx$$

$$= 2\pi \int_{-\pi/4}^{\pi/4} (1 + \cos 2x) \, dx - \pi [\tan x]_{-\pi/4}^{\pi/4} = 2\pi \left[x + \frac{1}{2} \sin 2x \right]_{-\pi/4}^{\pi/4} - \pi [1 - (-1)]$$

$$= 2\pi \left[\left(\frac{\pi}{4} + \frac{1}{2} \right) - \left(-\frac{\pi}{4} - \frac{1}{2} \right) \right] - 2\pi = 2\pi \left(\frac{\pi}{2} + 1 \right) - 2\pi = \pi^2$$

$$90. V = \int_{\pi/6}^{\pi/2} \pi \csc^2 x \, dx - \int_{\pi/6}^{\pi/2} \pi \sin^2 x \, dx = \pi \int_{\pi/6}^{\pi/2} \csc^2 x \, dx - \frac{\pi}{2} \int_{\pi/6}^{\pi/2} (1 - \cos 2x) \, dx$$

$$= \pi [-\cot x]_{\pi/6}^{\pi/2} - \frac{\pi}{2} \left[x - \frac{1}{2} \sin 2x \right]_{\pi/6}^{\pi/2} = \pi [0 - (-\sqrt{3})] - \frac{\pi}{2} \left[\left(\frac{\pi}{2} - 0 \right) - \left(\frac{\pi}{6} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \right) \right]$$

$$= \pi\sqrt{3} - \frac{\pi}{2} \left(\frac{2\pi}{6} + \frac{\sqrt{3}}{4} \right) = \pi \left(\frac{7\sqrt{3}}{8} - \frac{\pi}{6} \right)$$

$$\begin{aligned}
 91. \quad y = \ln(\cos x) &\Rightarrow \frac{dy}{dx} = -\frac{\sin x}{\cos x} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \tan^2 x = \sec^2 x - 1; \quad L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
 &= \int_0^{\pi/3} \sqrt{1 + (\sec^2 x - 1)} dx = \int_0^{\pi/3} \sec x dx = [\ln |\sec x + \tan x|]_0^{\pi/3} = \ln |2 + \sqrt{3}| - \ln |1 + 0| = \ln(2 + \sqrt{3})
 \end{aligned}$$

$$\begin{aligned}
 92. \quad y = \ln(\sec x) &\Rightarrow \frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \tan^2 x = \sec^2 x - 1; \quad L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
 &= \int_0^{\pi/4} \sec x dx = [\ln |\sec x + \tan x|]_0^{\pi/4} = \ln |\sqrt{2} + 1| - \ln |1 + 0| = \ln(\sqrt{2} + 1)
 \end{aligned}$$

$$\begin{aligned}
 93. \quad \int \csc x dx &= \int (\csc x)(1) dx = \int (\csc x) \left(\frac{\csc x + \cot x}{\csc x + \cot x} \right) dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx; \\
 \left[\begin{array}{l} u = \csc x + \cot x \\ du = (-\csc x \cot x - \csc^2 x) dx \end{array} \right] &\rightarrow \int \frac{-du}{u} = -\ln |u| + C = -\ln |\csc x + \cot x| + C
 \end{aligned}$$

$$\begin{aligned}
 94. \quad [(x^2 - 1)(x + 1)]^{-2/3} &= [(x - 1)(x + 1)^2]^{-2/3} = (x - 1)^{-2/3} (x + 1)^{-4/3} = (x + 1)^{-2} [(x - 1)^{-2/3} (x + 1)^{2/3}] \\
 &= (x + 1)^{-2} \left(\frac{x - 1}{x + 1} \right)^{-2/3} = (x + 1)^{-2} \left(1 - \frac{2}{x + 1} \right)^{-2/3}
 \end{aligned}$$

$$\begin{aligned}
 (a) \quad \int [(x^2 - 1)(x + 1)]^{-2/3} dx &= \int (x + 1)^{-2} \left(1 - \frac{2}{x + 1} \right)^{-2/3} dx; \quad \left[\begin{array}{l} u = \frac{1}{x + 1} \\ du = -\frac{1}{(x + 1)^2} dx \end{array} \right] \\
 &\rightarrow \int -(1 - 2u)^{-2/3} du = \frac{3}{2} (1 - 2u)^{1/3} + C = \frac{3}{2} \left(1 - \frac{2}{x + 1} \right)^{1/3} + C = \frac{3}{2} \left(\frac{x - 1}{x + 1} \right)^{1/3} + C
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \int [(x^2 - 1)(x + 1)]^{-2/3} dx &= \int (x + 1)^{-2} \left(\frac{x - 1}{x + 1} \right)^{-2/3} dx; \quad u = \left(\frac{x - 1}{x + 1} \right)^k \\
 \Rightarrow du &= k \left(\frac{x - 1}{x + 1} \right)^{k-1} \left[\frac{(x + 1) - (x - 1)}{(x + 1)^2} \right] dx = 2k \frac{(x - 1)^{k-1}}{(x + 1)^{k+1}} dx; \quad dx = \frac{(x + 1)^2}{2k} \left(\frac{x - 1}{x + 1} \right)^{k-1} du \\
 &= \frac{(x + 1)^2}{2k} \left(\frac{x - 1}{x + 1} \right)^{1-k} du; \quad \text{then, } \int \left(\frac{x - 1}{x + 1} \right)^{-2/3} \frac{1}{2k} \left(\frac{x - 1}{x + 1} \right)^{1-k} du = \frac{1}{2k} \int \left(\frac{x - 1}{x + 1} \right)^{(1/3-k)} du \\
 &= \frac{1}{2k} \int \left(\frac{x - 1}{x + 1} \right)^{k(1/3k-1)} du = \frac{1}{2k} \int u^{(1/3k-1)} du = \frac{1}{2k} (3k) u^{1/3k} + C = \frac{3}{2} u^{1/3k} + C = \frac{3}{2} \left(\frac{x - 1}{x + 1} \right)^{1/3} + C
 \end{aligned}$$

$$\begin{aligned}
 (c) \quad \int [(x^2 - 1)(x + 1)]^{-2/3} dx &= \int (x + 1)^{-2} \left(\frac{x - 1}{x + 1} \right)^{-2/3} dx; \\
 \left[\begin{array}{l} u = \tan^{-1} x \\ x = \tan u \\ dx = \frac{du}{\cos^2 u} \end{array} \right] &\rightarrow \int \frac{1}{(\tan u + 1)^2} \left(\frac{\tan u - 1}{\tan u + 1} \right)^{-2/3} \left(\frac{du}{\cos^2 u} \right) = \int \frac{1}{(\sin u + \cos u)^2} \left(\frac{\sin u - \cos u}{\sin u + \cos u} \right)^{-2/3} du;
 \end{aligned}$$

$$\left[\begin{array}{l} \sin u + \cos u = \sin u + \sin\left(\frac{\pi}{2} - u\right) = 2 \sin \frac{\pi}{4} \cos\left(u - \frac{\pi}{4}\right) \\ \sin u - \cos u = \sin u - \sin\left(\frac{\pi}{2} - u\right) = 2 \cos \frac{\pi}{4} \sin\left(u - \frac{\pi}{4}\right) \end{array} \right] \rightarrow \int \frac{1}{2 \cos^2\left(u - \frac{\pi}{4}\right)} \left[\frac{\sin\left(u - \frac{\pi}{4}\right)}{\cos\left(u - \frac{\pi}{4}\right)} \right]^{-2/3} du$$

$$= \frac{1}{2} \int \tan^{-2/3}\left(u - \frac{\pi}{4}\right) \sec^2\left(u - \frac{\pi}{4}\right) du = \frac{3}{2} \tan^{1/3}\left(u - \frac{\pi}{4}\right) + C = \frac{3}{2} \left[\frac{\tan u - \tan \frac{\pi}{4}}{1 + \tan u \tan \frac{\pi}{4}} \right]^{1/3} + C$$

$$= \frac{3}{2} \left(\frac{x-1}{x+1} \right)^{1/3} + C$$

$$(d) \quad u = \tan^{-1} \sqrt{x} \Rightarrow \tan u = \sqrt{x} \Rightarrow \tan^2 u = x \Rightarrow dx = 2 \tan u \left(\frac{1}{\cos^2 u} \right) du = \frac{2 \sin u}{\cos^3 u} du = -\frac{2d(\cos u)}{\cos^3 u};$$

$$x-1 = \tan^2 u - 1 = \frac{\sin^2 u - \cos^2 u}{\cos^2 u} = \frac{1-2\cos^2 u}{\cos^2 u}; \quad x+1 = \tan^2 u + 1 = \frac{\cos^2 u + \sin^2 u}{\cos^2 u} = \frac{1}{\cos^2 u};$$

$$\int (x-1)^{-2/3} (x+1)^{-4/3} dx = \int \frac{(1-2\cos^2 u)^{-2/3}}{(\cos^2 u)^{-2/3}} \cdot \frac{1}{(\cos^2 u)^{-4/3}} \cdot \frac{-2d(\cos u)}{\cos^3 u}$$

$$= \int (1-2\cos^2 u)^{-2/3} \cdot (-2) \cdot \cos u \cdot d(\cos u) = \frac{1}{2} \int (1-2\cos^2 u)^{-2/3} \cdot d(1-2\cos^2 u)$$

$$= \frac{3}{2} (1-2\cos^2 u)^{1/3} + C = \frac{3}{2} \left[\frac{\left(\frac{1-2\cos^2 u}{\cos^2 u} \right)^{1/3}}{\left(\frac{1}{\cos^2 u} \right)^{1/3}} \right] + C = \frac{3}{2} \left(\frac{x-1}{x+1} \right)^{1/3} + C$$

$$(e) \quad u = \tan^{-1} \left(\frac{x-1}{2} \right) \Rightarrow \frac{x-1}{2} = \tan u \Rightarrow x+1 = 2(\tan u + 1) \Rightarrow dx = \frac{2 du}{\cos^2 u} = 2d(\tan u);$$

$$\int (x-1)^{-2/3} (x+1)^{-4/3} dx = \int (\tan u)^{-2/3} (\tan u + 1)^{-4/3} \cdot 2^{-2} \cdot 2 \cdot d(\tan u)$$

$$= \frac{1}{2} \int \left(1 - \frac{1}{\tan u + 1} \right)^{-2/3} d\left(1 - \frac{1}{\tan u + 1} \right) = \frac{3}{2} \left(1 - \frac{1}{\tan u + 1} \right)^{1/3} + C = \frac{3}{2} \left(1 - \frac{2}{x+1} \right)^{1/3} + C$$

$$= \frac{3}{2} \left(\frac{x-1}{x+1} \right)^{1/3} + C$$

$$(f) \quad \left[\begin{array}{l} u = \cos^{-1} x \\ x = \cos u \\ dx = -\sin u du \end{array} \right] \rightarrow - \int \frac{\sin u du}{\sqrt[3]{(\cos^2 u - 1)^2} (\cos u + 1)^2} = - \int \frac{\sin u du}{(\sin^{4/3} u) \left(4 \cos \frac{u}{2} \right)^{4/3}}$$

$$= - \int \frac{du}{(\sin u)^{1/3} \left(4 \cos \frac{u}{2} \right)^{4/3}} = - \int \frac{du}{2 \left(\sin \frac{u}{2} \right)^{4/3} \left(\cos \frac{u}{2} \right)^{5/3}} = - \frac{1}{2} \int \left(\frac{\cos \frac{u}{2}}{\sin \frac{u}{2}} \right)^{1/3} \left(\frac{du}{\left(\cos^2 \frac{u}{2} \right)} \right)$$

$$= - \int \tan^{-1/3} \left(\frac{u}{2} \right) d\left(\tan \frac{u}{2} \right) = - \frac{3}{2} \tan^{2/3} \frac{u}{2} + C = \frac{3}{2} \left(-\tan^2 \frac{u}{2} \right)^{1/3} + C = \frac{3}{2} \left(\frac{\cos u - 1}{\cos u + 1} \right)^{1/3} + C$$

$$= \frac{3}{2} \left(\frac{x-1}{x+1} \right)^{1/3} + C$$

$$\begin{aligned}
 (g) \quad & \int \left[(x^2 - 1)(x + 1) \right]^{-2/3} dx; \left[\begin{array}{l} u = \cosh^{-1} x \\ x = \cosh u \\ dx = \sinh u \end{array} \right] \rightarrow \int \frac{\sinh u \, du}{\sqrt[3]{(\cosh^2 u - 1)^2 (\cosh u + 1)^2}} \\
 &= \int \frac{\sinh u \, du}{\sqrt[3]{(\sinh^4 u)(\cosh u + 1)^2}} = \int \frac{du}{\sqrt[3]{(\sinh u)(4 \cosh^4 \frac{u}{2})}} = \frac{1}{2} \int \frac{du}{\sqrt[3]{\sinh \left(\frac{u}{2} \right) \cosh^5 \left(\frac{u}{2} \right)}} \\
 &= \int \left(\tanh \frac{u}{2} \right)^{-1/3} d \left(\tanh \frac{u}{2} \right) = \frac{3}{2} \left(\tanh \frac{u}{2} \right)^{2/3} + C = \frac{3}{2} \left(\frac{\cosh x - 1}{\cosh x + 1} \right)^{1/3} + C = \frac{3}{2} \left(\frac{x - 1}{x + 1} \right)^{1/3} + C
 \end{aligned}$$

7.2 INTEGRATION BY PARTS

1. $u = x$, $du = dx$; $dv = \sin \frac{x}{2} \, dx$, $v = -2 \cos \frac{x}{2}$;

$$\int x \sin \frac{x}{2} \, dx = -2x \cos \frac{x}{2} - \int (-2 \cos \frac{x}{2}) \, dx = -2x \cos \left(\frac{x}{2} \right) + 4 \sin \left(\frac{x}{2} \right) + C$$

2. $u = \theta$, $du = d\theta$; $dv = \cos \pi\theta \, d\theta$, $v = \frac{1}{\pi} \sin \pi\theta$;

$$\int \theta \cos \pi\theta \, d\theta = \frac{\theta}{\pi} \sin \pi\theta - \int \frac{1}{\pi} \sin \pi\theta \, d\theta = \frac{\theta}{\pi} \sin \pi\theta + \frac{1}{\pi^2} \cos \pi\theta + C$$

3. $\cos t$

$$t^2 \xrightarrow{(+)} \sin t$$

$$2t \xrightarrow{(-)} -\cos t$$

$$2 \xrightarrow{(+)} -\sin t$$

0

$$\int t^2 \cos t \, dt = t^2 \sin t + 2t \cos t - 2 \sin t + C$$

4. $\sin x$

$$x^2 \xrightarrow{(+)} -\cos x$$

$$2x \xrightarrow{(-)} -\sin x$$

$$2 \xrightarrow{(+)} \cos x$$

0

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

5. $u = \ln x$, $du = \frac{dx}{x}$; $dv = x \, dx$, $v = \frac{x^2}{2}$;

$$\int_1^2 x \ln x \, dx = \left[\frac{x^2}{2} \ln x \right]_1^2 - \int_1^2 \frac{x^2}{2} \frac{dx}{x} = 2 \ln 2 - \left[\frac{x^2}{4} \right]_1^2 = 2 \ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}$$

6. $u = \ln x$, $du = \frac{dx}{x}$; $dv = x^3 dx$, $v = \frac{x^4}{4}$;

$$\int_1^e x^3 \ln x \, dx = \left[\frac{x^4}{4} \ln x \right]_1^e - \int_1^e \frac{x^4}{4} \frac{dx}{x} = \frac{e^4}{4} - \left[\frac{x^4}{16} \right]_1^e = \frac{3e^4 + 1}{16}$$

7. $u = \tan^{-1} y$, $du = \frac{dy}{1+y^2}$; $dv = dy$, $v = y$;

$$\int \tan^{-1} y \, dy = y \tan^{-1} y - \int \frac{y \, dy}{(1+y^2)} = y \tan^{-1} y - \frac{1}{2} \ln(1+y^2) + C = y \tan^{-1} y - \ln \sqrt{1+y^2} + C$$

8. $u = \sin^{-1} y$, $du = \frac{dy}{\sqrt{1-y^2}}$; $dv = dy$, $v = y$;

$$\int \sin^{-1} y \, dy = y \sin^{-1} y - \int \frac{y \, dy}{\sqrt{1-y^2}} = y \sin^{-1} y + \sqrt{1-y^2} + C$$

9. $u = x$, $du = dx$; $dv = \sec^2 x \, dx$, $v = \tan x$;

$$\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx = x \tan x + \ln |\cos x| + C$$

10. $\int 4x \sec^2 2x \, dx$; $[y = 2x] \rightarrow \int y \sec^2 y \, dy = y \tan y - \int \tan y \, dy = y \tan y - \ln |\sec y| + C$
 $= 2x \tan 2x - \ln |\sec 2x| + C$

11. e^x

$$x^3 \xrightarrow{(+)} e^x$$

$$3x^2 \xrightarrow{(-)} e^x$$

$$6x \xrightarrow{(+)} e^x$$

$$6 \xrightarrow{(-)} e^x$$

$$0$$

$$\int x^3 e^x \, dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C = (x^3 - 3x^2 + 6x - 6)e^x + C$$

12. e^{-p}

$$p^4 \xrightarrow{(+)} -e^{-p}$$

$$4p^3 \xrightarrow{(-)} e^{-p}$$

$$12p^2 \xrightarrow{(+)} -e^{-p}$$

$$24p \xrightarrow{(-)} e^{-p}$$

$$24 \xrightarrow{(+)} e^{-p}$$

$$0$$

$$\int p^4 e^{-p} \, dp = -p^4 e^{-p} - 4p^3 e^{-p} - 12p^2 e^{-p} - 24p e^{-p} - 24e^{-p} + C$$

$$= (-p^4 - 4p^3 - 12p^2 - 24p - 24)e^{-p} + C$$

$$\begin{array}{rcl}
 & & e^x \\
 13. & x^2 - 5x & \xrightarrow{(+)} e^x \\
 & 2x - 5 & \xrightarrow{(-)} e^x \\
 & 2 & \xrightarrow{(+)} e^x \\
 & 0 &
 \end{array}$$

$$\begin{aligned}
 \int (x^2 - 5x)e^x dx &= (x^2 - 5x)e^x - (2x - 5)e^x + 2e^x + C = x^2e^x - 7xe^x + 7e^x + C \\
 &= (x^2 - 7x + 7)e^x + C
 \end{aligned}$$

$$\begin{array}{rcl}
 & & e^r \\
 14. & r^2 + r + 1 & \xrightarrow{(+)} e^r \\
 & 2r + 1 & \xrightarrow{(-)} e^r \\
 & 2 & \xrightarrow{(+)} e^r \\
 & 0 &
 \end{array}$$

$$\begin{aligned}
 \int (r^2 + r + 1)e^r dr &= (r^2 + r + 1)e^r - (2r + 1)e^r + 2e^r + C \\
 &= [(r^2 + r + 1) - (2r + 1) + 2]e^r + C = (r^2 - r + 2)e^r + C
 \end{aligned}$$

$$\begin{array}{rcl}
 & & e^x \\
 15. & x^5 & \xrightarrow{(+)} e^x \\
 & 5x^4 & \xrightarrow{(-)} e^x \\
 & 20x^3 & \xrightarrow{(+)} e^x \\
 & 60x^2 & \xrightarrow{(-)} e^x \\
 & 120x & \xrightarrow{(+)} e^x \\
 & 120 & \xrightarrow{(-)} e^x \\
 & 0 &
 \end{array}$$

$$\begin{aligned}
 \int x^5 e^x dx &= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120x e^x - 120e^x + C \\
 &= (x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120)e^x + C
 \end{aligned}$$

$$\begin{array}{rcl}
 & & e^{4t} \\
 16. & t^2 & \xrightarrow{(+)} \frac{1}{4}e^{4t} \\
 & 2t & \xrightarrow{(-)} \frac{1}{16}e^{4t} \\
 & 2 & \xrightarrow{(+)} \frac{1}{64}e^{4t} \\
 & 0 &
 \end{array}$$

$$\begin{aligned}
 \int t^2 e^{4t} dt &= \frac{t^2}{4} e^{4t} - \frac{2t}{16} e^{4t} + \frac{2}{64} e^{4t} + C = \frac{t^2}{4} e^{4t} - \frac{t}{8} e^{4t} + \frac{1}{32} e^{4t} + C \\
 &= \left(\frac{t^2}{4} - \frac{t}{8} + \frac{1}{32} \right) e^{4t} + C
 \end{aligned}$$

$$\begin{array}{rcl}
 & & \sin 2\theta \\
 17. & \theta^2 & \xrightarrow{(+)} -\frac{1}{2} \cos 2\theta \\
 & 2\theta & \xrightarrow{(-)} -\frac{1}{4} \sin 2\theta
 \end{array}$$

$$\begin{aligned}
 & 2 \xrightarrow{(+)} \frac{1}{8} \cos 2\theta \\
 & 0 \quad \int_0^{\pi/2} \theta^2 \sin 2\theta \, d\theta = \left[-\frac{\theta^2}{2} \cos 2\theta + \frac{\theta}{2} \sin 2\theta + \frac{1}{4} \cos 2\theta \right]_0^{\pi/2} \\
 & \quad = \left[-\frac{\pi^2}{8} \cdot (-1) + \frac{\pi}{4} \cdot 0 + \frac{1}{4} \cdot (-1) \right] - \left[0 + 0 + \frac{1}{4} \cdot 1 \right] = \frac{\pi^2}{8} - \frac{1}{2} = \frac{\pi^2 - 4}{8}
 \end{aligned}$$

$$\begin{aligned}
 18. \quad & \cos 2x \\
 & x^3 \xrightarrow{(+)} \frac{1}{2} \sin 2x \\
 & 3x^2 \xrightarrow{(-)} -\frac{1}{4} \cos 2x \\
 & 6x \xrightarrow{(+)} -\frac{1}{8} \sin 2x \\
 & 6 \xrightarrow{(-)} \frac{1}{16} \cos 2x \\
 & 0 \quad \int_0^{\pi/2} x^3 \cos 2x \, dx = \left[\frac{x^3}{2} \sin 2x + \frac{3x^2}{4} \cos 2x - \frac{3x}{4} \sin 2x - \frac{3}{8} \cos 2x \right]_0^{\pi/2} \\
 & \quad = \left[\frac{\pi^3}{16} \cdot 0 + \frac{3\pi^2}{16} \cdot (-1) - \frac{3\pi}{8} \cdot 0 - \frac{3}{8} \cdot (-1) \right] - \left[0 + 0 - 0 - \frac{3}{8} \cdot 1 \right] = -\frac{3\pi^2}{16} + \frac{3}{4} = \frac{3(4 - \pi^2)}{16}
 \end{aligned}$$

$$\begin{aligned}
 19. \quad & u = \sec^{-1} t, \, du = \frac{dt}{t\sqrt{t^2-1}}; \, dv = t \, dt, \, v = \frac{t^2}{2}; \\
 & \int_{2/\sqrt{3}}^2 t \sec^{-1} t \, dt = \left[\frac{t^2}{2} \sec^{-1} t \right]_{2/\sqrt{3}}^2 - \int_{2/\sqrt{3}}^2 \left(\frac{t^2}{2} \right) \frac{dt}{t\sqrt{t^2-1}} = \left(2 \cdot \frac{\pi}{3} - \frac{2}{3} \cdot \frac{\pi}{6} \right) - \int_{2/\sqrt{3}}^2 \frac{t \, dt}{2\sqrt{t^2-1}} \\
 & = \frac{5\pi}{9} - \left[\frac{1}{2} \sqrt{t^2-1} \right]_{2/\sqrt{3}}^2 = \frac{5\pi}{9} - \frac{1}{2} \left(\sqrt{3} - \sqrt{\frac{4}{3}-1} \right) = \frac{5\pi}{9} - \frac{1}{2} \left(\sqrt{3} - \frac{\sqrt{3}}{3} \right) = \frac{5\pi}{9} - \frac{\sqrt{3}}{3} = \frac{5\pi - 3\sqrt{3}}{9}
 \end{aligned}$$

$$\begin{aligned}
 20. \quad & u = \sin^{-1}(x^2), \, du = \frac{2x \, dx}{\sqrt{1-x^4}}; \, dv = 2x \, dx, \, v = x^2; \\
 & \int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2) \, dx = \left[x^2 \sin^{-1}(x^2) \right]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} x^2 \cdot \frac{2x \, dx}{\sqrt{1-x^4}} = \left(\frac{1}{2} \right) \left(\frac{\pi}{6} \right) + \int_0^{1/\sqrt{2}} \frac{d(1-x^4)}{2\sqrt{1-x^4}} \\
 & = \frac{\pi}{12} + \left[\sqrt{1-x^4} \right]_0^{1/\sqrt{2}} = \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1 = \frac{\pi + 6\sqrt{3} - 12}{12}
 \end{aligned}$$

$$\begin{aligned}
 21. \quad & I = \int e^\theta \sin \theta \, d\theta; [u = \sin \theta, \, du = \cos \theta \, d\theta; \, dv = e^\theta \, d\theta, \, v = e^\theta] \Rightarrow I = e^\theta \sin \theta - \int e^\theta \cos \theta \, d\theta; \\
 & [u = \cos \theta, \, du = -\sin \theta \, d\theta; \, dv = e^\theta \, d\theta, \, v = e^\theta] \Rightarrow I = e^\theta \sin \theta - \left(e^\theta \cos \theta + \int e^\theta \sin \theta \, d\theta \right) \\
 & = e^\theta \sin \theta - e^\theta \cos \theta - I + C' \Rightarrow 2I = (e^\theta \sin \theta - e^\theta \cos \theta) + C' \Rightarrow I = \frac{1}{2}(e^\theta \sin \theta - e^\theta \cos \theta) + C, \text{ where } C = \frac{C'}{2} \text{ is} \\
 & \text{another arbitrary constant}
 \end{aligned}$$

$$\begin{aligned}
22. I &= \int e^{-y} \cos y \, dy; [u = \cos y, du = -\sin y \, dy; dv = e^{-y} \, dy, v = -e^{-y}] \\
&\Rightarrow I = -e^{-y} \cos y - \int (-e^{-y})(-\sin y) \, dy = -e^{-y} \cos y - \int e^{-y} \sin y \, dy; [u = \sin y, du = \cos y \, dy; \\
&dv = e^{-y} \, dy, v = -e^{-y}] \Rightarrow I = -e^{-y} \cos y - \left(-e^{-y} \sin y - \int (-e^{-y}) \cos y \, dy \right) = -e^{-y} \cos y + e^{-y} \sin y - I + C' \\
&\Rightarrow 2I = e^{-y}(\sin y - \cos y) + C' \Rightarrow I = \frac{1}{2}(e^{-y} \sin y - e^{-y} \cos y) + C, \text{ where } C = \frac{C'}{2} \text{ is another arbitrary constant}
\end{aligned}$$

$$\begin{aligned}
23. I &= \int e^{2x} \cos 3x \, dx; [u = \cos 3x; du = -3 \sin 3x \, dx, dv = e^{2x} \, dx; v = \frac{1}{2}e^{2x}] \\
&\Rightarrow I = \frac{1}{2}e^{2x} \cos 3x + \frac{3}{2} \int e^{2x} \sin 3x \, dx; [u = \sin 3x, du = 3 \cos 3x, dv = e^{2x} \, dx; v = \frac{1}{2}e^{2x}] \\
&\Rightarrow I = \frac{1}{2}e^{2x} \cos 3x + \frac{3}{2} \left(\frac{1}{2}e^{2x} \sin 3x - \frac{3}{2} \int e^{2x} \cos 3x \, dx \right) = \frac{1}{2}e^{2x} \cos 3x + \frac{3}{4}e^{2x} \sin 3x - \frac{9}{4}I + C' \\
&\Rightarrow \frac{13}{4}I = \frac{1}{2}e^{2x} \cos 3x + \frac{3}{4}e^{2x} \sin 3x + C' \Rightarrow \frac{e^{2x}}{13}(3 \sin 3x + 2 \cos 3x) + C, \text{ where } C = \frac{4}{13}C'
\end{aligned}$$

$$\begin{aligned}
24. \int e^{-2x} \sin 2x \, dx; [y = 2x] &\rightarrow \frac{1}{2} \int e^{-y} \sin y \, dy = I; [u = \sin y, du = \cos y \, dy; dv = e^{-y} \, dy, v = -e^{-y}] \\
&\Rightarrow I = \frac{1}{2} \left(-e^{-y} \sin y + \int e^{-y} \cos y \, dy \right) [u = \cos y, du = -\sin y; dv = e^{-y} \, dy, v = -e^{-y}] \\
&\Rightarrow I = -\frac{1}{2}e^{-y} \sin y + \frac{1}{2} \left(-e^{-y} \cos y - \int (-e^{-y})(-\sin y) \, dy \right) = -\frac{1}{2}e^{-y}(\sin y + \cos y) - I + C' \\
&\Rightarrow 2I = -\frac{1}{2}e^{-y}(\sin y + \cos y) + C' \Rightarrow I = -\frac{1}{4}e^{-y}(\sin y + \cos y) + C = -\frac{e^{-2x}}{4}(\sin 2x + \cos 2x) + C, \text{ where } \\
&C = \frac{C'}{2}
\end{aligned}$$

$$\begin{aligned}
25. \int e^{\sqrt{3s+9}} \, ds; \left[\frac{3s+9}{ds} = \frac{2}{3}x \, dx \right] &\rightarrow \int e^x \cdot \frac{2}{3}x \, dx = \frac{2}{3} \int x e^x \, dx; [u = x, du = dx; dv = e^x \, dx, v = e^x]; \\
\frac{2}{3} \int x e^x \, dx &= \frac{2}{3} \left(x e^x - \int e^x \, dx \right) = \frac{2}{3}(x e^x - e^x) + C = \frac{2}{3}(\sqrt{3s+9} e^{\sqrt{3s+9}} - e^{\sqrt{3s+9}}) + C
\end{aligned}$$

$$\begin{aligned}
26. u = x, du = dx; dv = \sqrt{1-x} \, dx, v &= -\frac{2}{3}\sqrt{(1-x)^3}; \\
\int_0^1 x \sqrt{1-x} \, dx &= \left[-\frac{2}{3}\sqrt{(1-x)^3} x \right]_0^1 + \frac{2}{3} \int_0^1 \sqrt{(1-x)^3} \, dx = \frac{2}{3} \left[-\frac{2}{5}(1-x)^{5/2} \right]_0^1 = \frac{4}{15}
\end{aligned}$$

$$\begin{aligned}
27. u = x, du = dx; dv = \tan^2 x \, dx, v &= \int \tan^2 x \, dx = \int \frac{\sin^2 x}{\cos^2 x} \, dx = \int \frac{1 - \cos^2 x}{\cos^2 x} \, dx = \int \frac{dx}{\cos^2 x} - \int dx \\
&= \tan x - x; \int_0^{\pi/3} x \tan^2 x \, dx = [x(\tan x - x)]_0^{\pi/3} - \int_0^{\pi/3} (\tan x - x) \, dx = \frac{\pi}{3} \left(\sqrt{3} - \frac{\pi}{3} \right) + \left[\ln |\cos x| + \frac{x^2}{2} \right]_0^{\pi/3}
\end{aligned}$$

$$= \frac{\pi}{3} \left(\sqrt{3} - \frac{\pi}{3} \right) + \ln \frac{1}{2} + \frac{\pi^2}{18} = \frac{\pi\sqrt{3}}{3} - \ln 2 - \frac{\pi^2}{18}$$

$$\begin{aligned} 28. \quad u = \ln(x+x^2), \quad du = \frac{(2x+1) dx}{x+x^2}; \quad dv = dx, \quad v = x; \quad \int \ln(x+x^2) dx = x \ln(x+x^2) - \int \frac{2x+1}{x(x+1)} \cdot x dx \\ = x \ln(x+x^2) - \int \frac{(2x+1) dx}{x+1} = x \ln(x+x^2) - \int \frac{2(x+1)-1}{x+1} dx = x \ln(x+x^2) - 2x + \ln|x+1| + C \end{aligned}$$

$$\begin{aligned} 29. \quad \int \sin(\ln x) dx; \quad \left[\begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \\ dx = e^u du \end{array} \right] \rightarrow \int (\sin u) e^u du. \quad \text{From Exercise 21, } \int (\sin u) e^u du = e^u \left(\frac{\sin u}{2} - \frac{\cos u}{2} \right) + C \\ = \frac{1}{2} [-x \cos(\ln x) + x \sin(\ln x)] + C \end{aligned}$$

$$30. \quad \int z(\ln z)^2 dz; \quad \left[\begin{array}{l} u = \ln z \\ du = \frac{1}{z} dz \\ dz = e^u du \end{array} \right] \rightarrow \int e^u \cdot u^2 \cdot e^u du = \int e^{2u} \cdot u^2 du;$$

$$u^2 \xrightarrow{(+)} \frac{1}{2} e^{2u}$$

$$2u \xrightarrow{(-)} \frac{1}{4} e^{2u}$$

$$2 \xrightarrow{(+)} \frac{1}{8} e^{2u}$$

0

$$\begin{aligned} \int u^2 e^{2u} du = \frac{u^2}{2} e^{2u} - \frac{u}{2} e^{2u} + \frac{1}{4} e^{2u} + C = \frac{e^{2u}}{4} [2u^2 - 2u + 1] + C \\ = \frac{z^2}{4} [2(\ln z)^2 - 2 \ln z + 1] + C \end{aligned}$$

$$31. \quad y = \int x^2 e^{4x} dx$$

$$\text{Let } u = x^2$$

$$dv = e^{4x} dx$$

$$du = 2x dx$$

$$v = \frac{1}{4} e^{4x}$$

$$y = (x^2) \left(\frac{1}{4} e^{4x} \right) - \int \left(\frac{1}{4} e^{4x} \right) (2x dx)$$

$$= \frac{1}{4} x^2 e^{4x} - \frac{1}{2} \int x e^{4x} dx$$

$$\text{Let } u = x$$

$$dv = e^{4x} dx$$

$$du = dx$$

$$v = \frac{1}{4} e^{4x}$$

$$y = \frac{1}{4} x^2 e^{4x} - \frac{1}{2} \left[(x) \left(\frac{1}{4} e^{4x} \right) - \int \left(\frac{1}{4} e^{4x} \right) dx \right]$$

$$y = \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x} + C$$

$$y = \left(\frac{x^2}{4} - \frac{x}{8} + \frac{1}{32}\right)e^{4x} + C$$

$$32. y = \int x^2 \ln x \, dx$$

$$\text{Let } u = \ln x \quad dv = x^2 \, dx$$

$$du = \frac{1}{x} \, dx \quad v = \frac{1}{3}x^3$$

$$y = (\ln x)\left(\frac{1}{3}x^3\right) - \int \left(\frac{1}{3}x^3\right)\left(\frac{1}{x} \, dx\right)$$

$$y = \frac{1}{3}x^3 \ln x - \frac{1}{3} \int x^2 \, dx$$

$$y = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$$

$$33. \text{ Let } w = \sqrt{\theta}. \text{ Then } dw = \frac{d\theta}{2\sqrt{\theta}}, \text{ so } d\theta = 2\sqrt{\theta} \, dw = 2w \, dw.$$

$$\int \sin \sqrt{\theta} \, d\theta = \int (\sin w)(2w \, dw) = 2 \int w \sin w \, dw$$

$$\text{Let } u = w \quad dv = \sin w \, dw$$

$$du = dw \quad v = -\cos w$$

$$\begin{aligned} \int w \sin w \, dw &= -w \cos w + \int \cos w \, dw \\ &= -w \cos w + \sin w + C \end{aligned}$$

$$\begin{aligned} \int \sin \sqrt{\theta} \, d\theta &= 2 \int w \sin w \, dw \\ &= -2w \cos w + 2 \sin w + C \\ &= -2\sqrt{\theta} \cos \sqrt{\theta} + 2 \sin \sqrt{\theta} + C \end{aligned}$$

$$34. y = \int \theta \sec \theta \tan \theta \, d\theta$$

$$\text{Let } u = \theta \quad dv = \sec \theta \tan \theta \, d\theta$$

$$du = d\theta \quad v = \sec \theta$$

$$y = \theta \sec \theta - \int \sec \theta \, d\theta$$

$$y = \theta \sec \theta - \ln |\sec \theta + \tan \theta| + C$$

$$35. (a) \, u = x, \, du = dx; \, dv = \sin x \, dx, \, v = -\cos x;$$

$$S_1 = \int_0^\pi x \sin x \, dx = [-x \cos x]_0^\pi + \int_0^\pi \cos x \, dx = \pi + [\sin x]_0^\pi = \pi$$

$$(b) S_2 = - \int_{\pi}^{2\pi} x \sin x \, dx = \left[-x \cos x \right]_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \cos x \, dx = -[-3\pi + [\sin x]_{\pi}^{2\pi}] = 3\pi$$

$$(c) S_3 = \int_{2\pi}^{3\pi} x \sin x \, dx = [-x \cos x]_{2\pi}^{3\pi} + \int_{2\pi}^{3\pi} \cos x \, dx = 5\pi + [\sin x]_{2\pi}^{3\pi} = 5\pi$$

$$(d) S_{n+1} = (-1)^{n+1} \int_{n\pi}^{(n+1)\pi} x \sin x \, dx = (-1)^{n+1} \left[-x \cos x \right]_{n\pi}^{(n+1)\pi} + [\sin x]_{n\pi}^{(n+1)\pi} \\ = (-1)^{n+1} [-(n+1)\pi(-1)^n + n\pi(-1)^{n+1}] + 0 = (2n+1)\pi$$

36. (a) $u = x$, $du = dx$; $dv = \cos x \, dx$, $v = \sin x$;

$$S_1 = - \int_{\pi/2}^{3\pi/2} x \cos x \, dx = - \left[x \sin x \right]_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} \sin x \, dx = - \left(-\frac{3\pi}{2} - \frac{\pi}{2} \right) - [\cos x]_{\pi/2}^{3\pi/2} = 2\pi$$

$$(b) S_2 = \int_{3\pi/2}^{5\pi/2} x \cos x \, dx = [x \sin x]_{3\pi/2}^{5\pi/2} - \int_{3\pi/2}^{5\pi/2} \sin x \, dx = \left[\frac{5\pi}{2} - \left(-\frac{3\pi}{2} \right) \right] - [\cos x]_{3\pi/2}^{5\pi/2} = 4\pi$$

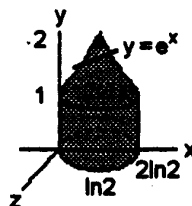
$$(c) S_3 = - \int_{5\pi/2}^{7\pi/2} x \cos x \, dx = - \left[x \sin x \right]_{5\pi/2}^{7\pi/2} - \int_{5\pi/2}^{7\pi/2} \sin x \, dx = - \left(-\frac{7\pi}{2} - \frac{5\pi}{2} \right) - [\cos x]_{5\pi/2}^{7\pi/2} = 6\pi$$

$$(d) S_n = (-1)^n \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} x \cos x \, dx = (-1)^n \left[x \sin x \right]_{(2n-1)\pi/2}^{(2n+1)\pi/2} - \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} \sin x \, dx \\ = (-1)^n \left[\frac{(2n+1)\pi}{2} (-1)^n - \frac{(2n-1)\pi}{2} (-1)^{n-1} \right] - [\cos x]_{(2n-1)\pi/2}^{(2n+1)\pi/2} = \frac{1}{2} (2n\pi + \pi + 2n\pi - \pi) = 2n\pi$$

$$37. V = \int_0^{\ln 2} 2\pi(\ln 2 - x) e^x \, dx = 2\pi \ln 2 \int_0^{\ln 2} e^x \, dx - 2\pi \int_0^{\ln 2} x e^x \, dx$$

$$= (2\pi \ln 2) [e^x]_0^{\ln 2} - 2\pi \left([x e^x]_0^{\ln 2} - \int_0^{\ln 2} e^x \, dx \right)$$

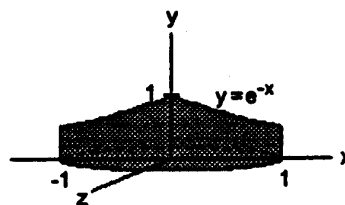
$$= 2\pi \ln 2 - 2\pi (2 \ln 2 + [e^x]_0^{\ln 2}) = -2\pi \ln 2 + 2 = 2\pi(1 - \ln 2)$$



$$38. (a) V = \int_0^1 2\pi x e^{-x} \, dx = 2\pi \left([-x e^{-x}]_0^1 + \int_0^1 e^{-x} \, dx \right)$$

$$= 2\pi \left(-\frac{1}{e} + [-e^{-x}]_0^1 \right) = 2\pi \left(-\frac{1}{e} - \frac{1}{e} + 1 \right)$$

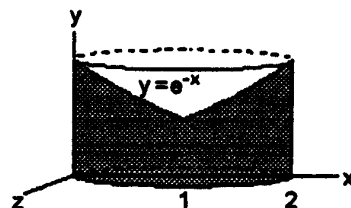
$$= 2\pi - \frac{4\pi}{e}$$



$$(b) V = \int_0^1 2\pi(1-x)e^{-x} dx; u = 1-x, du = -dx; dv = e^{-x} dx,$$

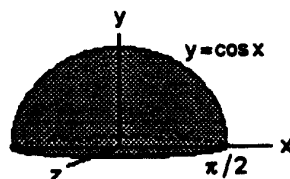
$$v = -e^{-x}; V = 2\pi \left[(1-x)(-e^{-x}) \right]_0^1 - \int_0^1 e^{-x} dx$$

$$= 2\pi \left[[0 - 1(-1)] + [e^{-x}]_0^1 \right] = 2\pi \left(1 + \frac{1}{e} - 1 \right) = \frac{2\pi}{e}$$



$$39. (a) V = \int_0^{\pi/2} 2\pi x \cos x dx = 2\pi \left([x \sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x dx \right)$$

$$= 2\pi \left(\frac{\pi}{2} + [\cos x]_0^{\pi/2} \right) = 2\pi \left(\frac{\pi}{2} + 0 - 1 \right) = \pi(\pi - 2)$$



$$(b) V = \int_0^{\pi/2} 2\pi \left(\frac{\pi}{2} - x \right) \cos x dx; u = \frac{\pi}{2} - x, du = -dx; dv = \cos x dx, v = \sin x;$$

$$V = 2\pi \left[\left(\frac{\pi}{2} - x \right) \sin x \right]_0^{\pi/2} + 2\pi \int_0^{\pi/2} \sin x dx = 0 + 2\pi [-\cos x]_0^{\pi/2} = 2\pi(0 + 1) = 2\pi$$

$$40. (a) V = \int_0^{\pi} 2\pi x(x \sin x) dx;$$

$$\begin{array}{rcl} & \sin x & \\ x^2 & \xrightarrow{(+)} & -\cos x \\ 2x & \xrightarrow{(-)} & -\sin x \\ 2 & \xrightarrow{(+)} & \cos x \\ 0 & & \end{array}$$

$$\Rightarrow V = 2\pi \int_0^{\pi} x^2 \sin x dx = 2\pi [-x^2 \cos x + 2x \sin x + 2 \cos x]_0^{\pi} = 2\pi(\pi^2 - 4)$$

$$(b) V = \int_0^{\pi} 2\pi(\pi - x)x \sin x dx = 2\pi^2 \int_0^{\pi} x \sin x dx - 2\pi \int_0^{\pi} x^2 \sin x dx = 2\pi^2 [-x \cos x + \sin x]_0^{\pi} - (2\pi^3 - 8\pi)$$

$$= 8\pi$$

$$41. av(y) = \frac{1}{2\pi} \int_0^{2\pi} 2e^{-t} \cos t dt = \frac{1}{\pi} \left[e^{-t} \left(\frac{\sin t - \cos t}{2} \right) \right]_0^{2\pi}$$

$$(\text{see Exercise 22}) \Rightarrow av(y) = \frac{1}{2\pi} (1 - e^{-2\pi})$$

$$42. av(y) = \frac{1}{2\pi} \int_0^{2\pi} 4e^{-t} (\sin t - \cos t) dt = \frac{2}{\pi} \int_0^{2\pi} e^{-t} \sin t dt - \frac{2}{\pi} \int_0^{2\pi} e^{-t} \cos t dt$$

$$= \frac{2}{\pi} \left[e^{-t} \left(\frac{-\sin t - \cos t}{2} \right) - e^{-t} \left(\frac{\sin t - \cos t}{2} \right) \right]_0^{2\pi} = \frac{2}{\pi} [-e^{-t} \sin t]_0^{2\pi} = 0$$

43. Let $u = x^n$ $dv = \cos x \, dx$

$du = nx^{n-1} \, dx$ $v = \sin x$

$$\int x^n \cos x \, dx = x^n \sin x - \int (\sin x)(nx^{n-1} \, dx) = x^n \sin x - n \int x^{n-1} \sin x \, dx$$

44. Let $u = x^n$ $dv = \sin x \, dx$

$du = nx^{n-1} \, dx$ $v = -\cos x$

$$\int x^n \sin x \, dx = (x^n)(-\cos x) - \int (-\cos x)(nx^{n-1} \, dx) = -x^n \cos x + n \int x^{n-1} \cos x \, dx$$

45. Let $u = x^n$ $dv = e^{ax} \, dx$

$du = nx^{n-1} \, dx$ $v = \frac{1}{a} e^{ax}$

$$\int x^n e^{ax} \, dx = (x^n)\left(\frac{1}{a} e^{ax}\right) - \int \left(\frac{1}{a} e^{ax}\right)(nx^{n-1} \, dx) = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx, a \neq 0$$

46. Let $u = (\ln x)^n$ $dv = dx$

$du = \frac{n(\ln x)^{n-1}}{x} \, dx$ $v = x$

$$\int (\ln x)^n \, dx = x(\ln x)^n - \int x \left[\frac{n(\ln x)^{n-1}}{x} \right] dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx$$

47. (a) Let $y = f^{-1}(x)$. Then $x = f(y)$, so $dx = f'(y) \, dy$.

Hence, $\int f^{-1}(x) \, dx = \int (y)[f'(y) \, dy] = \int yf'(y) \, dy$

(b) Let $u = y$ $dv = f'(y) \, dy$

$du = dy$ $v = f(y)$

$$\int yf'(y) \, dy = yf(y) - \int f(y) \, dy = f^{-1}(x)(x) - \int f(y) \, dy$$

Hence, $\int f^{-1}(x) \, dx = \int yf'(y) \, dy = xf^{-1}(x) - \int f(y) \, dy.$

48. Let $u = f^{-1}(x)$ $dv = dx$

$du = \left(\frac{d}{dx} f^{-1}(x)\right) dx$ $v = x$

$$\int f^{-1}(x) \, dx = xf^{-1}(x) - \int x \left(\frac{d}{dx} f^{-1}(x)\right) dx$$

49. (a) Using $y = f^{-1}(x) = \sin^{-1} x$ and $f(y) = \sin y$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, we have:

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \sin y \, dy = x \sin^{-1} x + \cos y + C = x \sin^{-1} x + \cos(\sin^{-1} x) + C$$

$$(b) \int \sin^{-1} x \, dx = x \sin^{-1} x - \int x \left(\frac{d}{dx} \sin^{-1} x \right) dx = x \sin^{-1} x - \int x \frac{1}{\sqrt{1-x^2}} dx$$

$$u = 1 - x^2, \, du = -2x \, dx = x \sin^{-1} x + \frac{1}{2} \int u^{-1/2} du = x \sin^{-1} x + u^{1/2} + C = x \sin^{-1} x + \sqrt{1-x^2} + C$$

$$(c) \cos(\sin^{-1} x) = \sqrt{1-x^2}$$

50. (a) Using $y = f^{-1}(x) = \tan^{-1} x$ and $f(y) = \tan y$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$, we have:

$$\begin{aligned} \int \tan^{-1} x \, dx &= x \tan^{-1} x - \int \tan y \, dy = x \tan^{-1} x - \ln |\sec y| + C = x \tan^{-1} x + \ln |\cos y| + C \\ &= x \tan^{-1} x + \ln |\cos(\tan^{-1}(x))| \, dx + C \end{aligned}$$

$$(b) \int \tan^{-1} x \, dx = x \tan^{-1} x - \int x \left(\frac{d}{dx} \tan^{-1} x \right) dx = x \tan^{-1} x - \int x \left(\frac{1}{1+x^2} \right) dx$$

$$u = 1 + x^2, \, du = 2x \, dx = x \tan^{-1} x - \frac{1}{2} \int u^{-1} du = x \tan^{-1} x - \frac{1}{2} \ln |u| + C = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$$

$$(c) \ln |\cos(\tan^{-1} x)| = \ln \left| \frac{1}{\sqrt{1+x^2}} \right| = -\frac{1}{2} \ln(1+x^2)$$

51. (a) Using $y = f^{-1}(x) = \cos^{-1} x$ and $f(y) = \cos x$, $0 \leq x \leq \pi$, we have:

$$\int \cos^{-1} x \, dx = x \cos^{-1} x - \int \cos y \, dy = x \cos^{-1} x - \sin y + C = x \cos^{-1} x - \sin(\cos^{-1} x) + C$$

$$(b) \int \cos^{-1} x \, dx = x \cos^{-1} x - \int x \left(\frac{d}{dx} \cos^{-1} x \right) dx = x \cos^{-1} x - \int x \left(-\frac{1}{\sqrt{1-x^2}} \right) dx$$

$$u = 1 - x^2, \, du = -2x \, dx = x \cos^{-1} x - \frac{1}{2} \int u^{-1/2} du = x \cos^{-1} x - u^{1/2} + C = x \cos^{-1} x - \sqrt{1-x^2} + C$$

$$(c) \sin(\cos^{-1} x) = \sqrt{1-x^2}$$

52. (a) Using $y = f^{-1}(x) = \log_2 x$ and $f(y) = 2^y$, we have

$$\int \log_2 x \, dx = x \log_2 x - \int 2^y \, dy = x \log_2 x - \frac{2^y}{\ln 2} + C = x \log_2 x - \frac{1}{\ln 2} 2^{\log_2 x}$$

$$(b) \int \log_2 x \, dx = x \log_2 x - \int x \left(\frac{d}{dx} \log_2 x \right) dx = x \log_2 x - \int x \left(\frac{1}{x \ln 2} \right) dx = x \log_2 x - \int \frac{dx}{\ln 2}$$

$$= x \log_2 x - \left(\frac{1}{\ln 2} \right) x + C$$

$$(c) 2^{\log_2 x} = x$$

7.3 PARTIAL FRACTIONS

$$1. \frac{5x-13}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} \Rightarrow 5x-13 = A(x-2) + B(x-3) = (A+B)x - (2A+3B)$$

$$\Rightarrow \begin{cases} A+B=5 \\ 2A+3B=13 \end{cases} \Rightarrow -B = (10-13) \Rightarrow B=3 \Rightarrow A=2; \text{ thus, } \frac{5x-13}{(x-3)(x-2)} = \frac{2}{x-3} + \frac{3}{x-2}$$

$$2. \frac{5x-7}{x^2-3x+2} = \frac{5x-7}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 5x-7 = A(x-1) + B(x-2) = (A+B)x - (A+2B)$$

$$\Rightarrow \begin{cases} A+B=5 \\ A+2B=7 \end{cases} \Rightarrow B=2 \Rightarrow A=3; \text{ thus, } \frac{5x-7}{x^2-3x+2} = \frac{3}{x-2} + \frac{2}{x-1}$$

$$3. \frac{x+4}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow x+4 = A(x+1) + B = Ax + (A+B) \Rightarrow \begin{cases} A=1 \\ A+B=4 \end{cases} \Rightarrow A=1 \text{ and } B=3;$$

$$\text{thus, } \frac{x+4}{(x+1)^2} = \frac{1}{x+1} + \frac{3}{(x+1)^2}$$

$$4. \frac{2x+2}{x^2-2x+1} = \frac{2x+2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 2x+2 = A(x-1) + B = Ax + (-A+B) \Rightarrow \begin{cases} A=2 \\ -A+B=2 \end{cases}$$

$$\Rightarrow A=2 \text{ and } B=4; \text{ thus, } \frac{2x+2}{x^2-2x+1} = \frac{2}{x-1} + \frac{4}{(x-1)^2}$$

$$5. \frac{z+1}{z^2(z-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} \Rightarrow z+1 = Az(z-1) + B(z-1) + Cz^2 \Rightarrow z+1 = (A+C)z^2 + (-A+B)z - B$$

$$\Rightarrow \begin{cases} A+C=0 \\ -A+B=1 \\ -B=1 \end{cases} \Rightarrow B=-1 \Rightarrow A=-2 \Rightarrow C=2; \text{ thus, } \frac{z+1}{z^2(z-1)} = \frac{-2}{z} + \frac{-1}{z^2} + \frac{2}{z-1}$$

$$6. \frac{z}{z^3-z^2-6z} = \frac{1}{z^2-z-6} = \frac{1}{(z-3)(z+2)} = \frac{A}{z-3} + \frac{B}{z+2} \Rightarrow 1 = A(z+2) + B(z-3) = (A+B)z + (2A-3B)$$

$$\Rightarrow \begin{cases} A+B=0 \\ 2A-3B=1 \end{cases} \Rightarrow -5B=1 \Rightarrow B=-\frac{1}{5} \Rightarrow A=\frac{1}{5}; \text{ thus, } \frac{z}{z^3-z^2-6z} = \frac{\frac{1}{5}}{z-3} + \frac{-\frac{1}{5}}{z+2}$$

$$7. \frac{t^2+8}{t^2-5t+6} = 1 + \frac{5t+2}{t^2-5t+6} \text{ (after long division); } \frac{5t+2}{t^2-5t+6} = \frac{5t+2}{(t-3)(t-2)} = \frac{A}{t-3} + \frac{B}{t-2}$$

$$\Rightarrow 5t+2 = A(t-2) + B(t-3) = (A+B)t + (-2A-3B) \Rightarrow \begin{cases} A+B=5 \\ -2A-3B=2 \end{cases} \Rightarrow -B=(10+2)=12$$

$$\Rightarrow B=-12 \Rightarrow A=17; \text{ thus, } \frac{t^2+8}{t^2-5t+6} = 1 + \frac{17}{t-3} + \frac{-12}{t-2}$$

$$8. \frac{t^4+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^2(t^2+9)} \text{ (after long division); } \frac{-9t^2+9}{t^2(t^2+9)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+9}$$

$$\Rightarrow -9t^2+9 = At(t^2+9) + B(t^2+9) + (Ct+D)t^2 = (A+C)t^3 + (B+D)t^2 + 9At + 9B$$

$$\Rightarrow \begin{cases} A+C=0 \\ B+D=-9 \\ 9A=0 \\ 9B=9 \end{cases} \Rightarrow A=0 \Rightarrow C=0; B=1 \Rightarrow D=-10; \text{ thus, } \frac{t^4+9}{t^4+9t^2} = 1 + \frac{1}{t^2} + \frac{-10}{t^2+9}$$

$$9. \frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \Rightarrow 1 = A(1+x) + B(1-x); x=1 \Rightarrow A=\frac{1}{2}; x=-1 \Rightarrow B=\frac{1}{2};$$

$$\int \frac{dx}{1-x^2} = \frac{1}{2} \int \frac{dx}{1-x} + \frac{1}{2} \int \frac{dx}{1+x} = \frac{1}{2} [\ln|1+x| - \ln|1-x|] + C$$

$$10. \frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow 1 = A(x+2) + Bx; x=0 \Rightarrow A = \frac{1}{2}; x=-2 \Rightarrow B = -\frac{1}{2};$$

$$\int \frac{dx}{x^2+2x} = \frac{1}{2} \int \frac{dx}{x} - \frac{1}{2} \int \frac{dx}{x+2} = \frac{1}{2} [\ln|x| - \ln|x+2|] + C$$

$$11. \frac{x+4}{x^2+5x-6} = \frac{A}{x+6} + \frac{B}{x-1} \Rightarrow x+4 = A(x-1) + B(x+6); x=1 \Rightarrow B = \frac{5}{7}; x=-6 \Rightarrow A = \frac{-2}{-7} = \frac{2}{7};$$

$$\int \frac{x+4}{x^2+5x-6} dx = \frac{2}{7} \int \frac{dx}{x+6} + \frac{5}{7} \int \frac{dx}{x-1} = \frac{2}{7} \ln|x+6| + \frac{5}{7} \ln|x-1| + C = \frac{1}{7} \ln|(x+6)^2(x-1)^5| + C$$

$$12. \frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3} \Rightarrow 2x+1 = A(x-3) + B(x-4); x=3 \Rightarrow B = \frac{7}{-1} = -7; x=4 \Rightarrow A = \frac{9}{1} = 9;$$

$$\int \frac{2x+1}{x^2-7x+12} dx = 9 \int \frac{dx}{x-4} - 7 \int \frac{dx}{x-3} = 9 \ln|x-4| - 7 \ln|x-3| + C = \ln \left| \frac{(x-4)^9}{(x-3)^7} \right| + C$$

$$13. \frac{y}{y^2-2y-3} = \frac{A}{y-3} + \frac{B}{y+1} \Rightarrow y = A(y+1) + B(y-3); y=-1 \Rightarrow B = \frac{-1}{-4} = \frac{1}{4}; y=3 \Rightarrow A = \frac{3}{4};$$

$$\begin{aligned} \int_4^8 \frac{y}{y^2-2y-3} dy &= \frac{3}{4} \int_4^8 \frac{dy}{y-3} + \frac{1}{4} \int_4^8 \frac{dy}{y+1} = \left[\frac{3}{4} \ln|y-3| + \frac{1}{4} \ln|y+1| \right]_4^8 = \left(\frac{3}{4} \ln 5 + \frac{1}{4} \ln 9 \right) - \left(\frac{3}{4} \ln 1 + \frac{1}{4} \ln 5 \right) \\ &= \frac{1}{2} \ln 5 + \frac{1}{2} \ln 3 = \frac{\ln 15}{2} \end{aligned}$$

$$14. \frac{y+4}{y^2+y} = \frac{A}{y} + \frac{B}{y+1} \Rightarrow y+4 = A(y+1) + By; y=0 \Rightarrow A = 4; y=-1 \Rightarrow B = \frac{3}{-1} = -3;$$

$$\begin{aligned} \int_{1/2}^1 \frac{y+4}{y^2+y} dy &= 4 \int_{1/2}^1 \frac{dy}{y} - 3 \int_{1/2}^1 \frac{dy}{y+1} = [4 \ln|y| - 3 \ln|y+1|]_{1/2}^1 = (4 \ln 1 - 3 \ln 2) - \left(4 \ln \frac{1}{2} - 3 \ln \frac{3}{2} \right) \\ &= \ln \frac{1}{8} - \ln \frac{1}{16} + \ln \frac{27}{8} = \ln \left(\frac{27}{8} \cdot \frac{1}{8} \cdot 16 \right) = \ln \frac{27}{4} \end{aligned}$$

$$15. \frac{1}{t^3+t^2-2t} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1} \Rightarrow 1 = A(t+2)(t-1) + Bt(t-1) + Ct(t+2); t=0 \Rightarrow A = -\frac{1}{2}; t=-2$$

$$\begin{aligned} \Rightarrow B = \frac{1}{6}; t=1 \Rightarrow C = \frac{1}{3}; \int \frac{dt}{t^3+t^2-2t} &= -\frac{1}{2} \int \frac{dt}{t} + \frac{1}{6} \int \frac{dt}{t+2} + \frac{1}{3} \int \frac{dt}{t-1} \\ &= -\frac{1}{2} \ln|t| + \frac{1}{6} \ln|t+2| + \frac{1}{3} \ln|t-1| + C \end{aligned}$$

$$16. \frac{x+3}{2x^3-8x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} \Rightarrow \frac{1}{2}(x+3) = A(x+2)(x-2) + Bx(x-2) + Cx(x+2); x=0 \Rightarrow A = \frac{3}{-8}; x=-2$$

$$\begin{aligned} \Rightarrow B = \frac{1}{16}; x=2 \Rightarrow C = \frac{5}{16}; \int \frac{x+3}{2x^3-8x} dx &= -\frac{3}{8} \int \frac{dx}{x} + \frac{1}{16} \int \frac{dx}{x+2} + \frac{5}{16} \int \frac{dx}{x-2} \\ &= -\frac{3}{8} \ln|x| + \frac{1}{16} \ln|x+2| + \frac{5}{16} \ln|x-2| + C = \frac{1}{16} \ln \left| \frac{(x-2)^5(x+2)}{x^6} \right| + C \end{aligned}$$

$$17. \frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{(x+1)^2} \text{ (after long division); } \frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 3x+2 = A(x+1) + B$$

$$= Ax + (A+B) \Rightarrow A=3, A+B=2 \Rightarrow A=3, B=-1; \int_0^1 \frac{x^3 dx}{x^2+2x+1}$$

$$= \int_0^1 (x-2) dx + 3 \int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2} = \left[\frac{x^2}{2} - 2x + 3 \ln|x+1| + \frac{1}{x+1} \right]_0^1$$

$$= \left(\frac{1}{2} - 2 + 3 \ln 2 + \frac{1}{2} \right) - (1) = 3 \ln 2 - 2$$

$$18. \frac{x^3}{x^2-2x+1} = (x+2) + \frac{3x+2}{(x-1)^2} \text{ (after long division); } \frac{3x+2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 3x+2 = A(x-1) + B$$

$$= Ax + (-A+B) \Rightarrow A=3, -A+B=-2 \Rightarrow A=3, B=1; \int_{-1}^0 \frac{x^3 dx}{x^2-2x+1}$$

$$= \int_{-1}^0 (x+2) dx + 3 \int_{-1}^0 \frac{dx}{x-1} + \int_{-1}^0 \frac{dx}{(x-1)^2} = \left[\frac{x^2}{2} + 2x + 3 \ln|x-1| - \frac{1}{x-1} \right]_{-1}^0$$

$$= \left(0 + 0 + 3 \ln 1 - \frac{1}{(-1)} \right) - \left(\frac{1}{2} - 2 + 3 \ln 2 - \frac{1}{(-2)} \right) = 2 - 3 \ln 2$$

$$19. \frac{1}{(x^2-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x+1)^2} + \frac{D}{(x-1)^2} \Rightarrow 1 = A(x+1)(x-1)^2 + B(x-1)(x+1)^2 + C(x-1)^2 + D(x+1)^2;$$

$$x=-1 \Rightarrow C=\frac{1}{4}; x=1 \Rightarrow D=\frac{1}{4}; \text{coefficient of } x^3 = A+B \Rightarrow A+B=0; \text{constant} = A-B+C+D$$

$$\Rightarrow A-B+C+D=1 \Rightarrow A-B=\frac{1}{2}; \text{thus, } A=\frac{1}{4} \Rightarrow B=-\frac{1}{4}; \int \frac{dx}{(x^2-1)^2}$$

$$= \frac{1}{4} \int \frac{dx}{x+1} - \frac{1}{4} \int \frac{dx}{x-1} + \frac{1}{4} \int \frac{dx}{(x+1)^2} + \frac{1}{4} \int \frac{dx}{(x-1)^2} = \frac{1}{4} \ln \left| \frac{x+1}{x-1} \right| - \frac{x}{2(x^2-1)} + C$$

$$20. \frac{x^2}{(x-1)(x^2+2x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \Rightarrow x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1); x=-1$$

$$\Rightarrow C=-\frac{1}{2}; x=1 \Rightarrow A=\frac{1}{4}; \text{coefficient of } x^2 = A+B \Rightarrow A+B=1 \Rightarrow B=\frac{3}{4}; \int \frac{x^2 dx}{(x-1)(x^2+2x+1)}$$

$$= \frac{1}{4} \int \frac{dx}{x-1} + \frac{3}{4} \int \frac{dx}{x+1} - \frac{1}{2} \int \frac{dx}{(x+1)^2} = \frac{1}{4} \ln|x-1| + \frac{3}{4} \ln|x+1| + \frac{1}{2(x+1)} + C$$

$$= \frac{\ln|(x-1)(x+1)^3|}{4} + \frac{1}{2(x+1)} + C$$

$$21. \frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \Rightarrow 1 = A(x^2+1) + (Bx+C)(x+1); x=-1 \Rightarrow A=\frac{1}{2}; \text{coefficient of } x^2$$

$$= A+B \Rightarrow A+B=0 \Rightarrow B=-\frac{1}{2}; \text{constant} = A+C \Rightarrow A+C=1 \Rightarrow C=\frac{1}{2}; \int_0^1 \frac{dx}{(x+1)(x^2+1)}$$

$$\begin{aligned}
&= \frac{1}{2} \int_0^1 \frac{dx}{x+1} + \frac{1}{2} \int_0^1 \frac{(-x+1)}{x^2+1} dx = \left[\frac{1}{2} \ln|x+1| - \frac{1}{4} \ln(x^2+1) + \frac{1}{2} \tan^{-1} x \right]_0^1 \\
&= \left(\frac{1}{2} \ln 2 - \frac{1}{4} \ln 2 + \frac{1}{2} \tan^{-1} 1 \right) - \left(\frac{1}{2} \ln 1 - \frac{1}{4} \ln 1 + \frac{1}{2} \tan^{-1} 0 \right) = \frac{1}{4} \ln 2 + \frac{1}{2} \left(\frac{\pi}{4} \right) = \frac{(\pi + 2 \ln 2)}{8}
\end{aligned}$$

$$22. \frac{3t^2+t+4}{t^3+t} = \frac{A}{t} + \frac{Bt+C}{t^2+1} \Rightarrow 3t^2+t+4 = A(t^2+1) + (Bt+C)t; t=0 \Rightarrow A=4; \text{coefficient of } t^2$$

$$= A+B \Rightarrow A+B=3 \Rightarrow B=-1; \text{coefficient of } t=C \Rightarrow C=1; \int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t} dt$$

$$= 4 \int_1^{\sqrt{3}} \frac{dt}{t} + \int_1^{\sqrt{3}} \frac{(-t+1)}{t^2+1} dt = \left[4 \ln|t| - \frac{1}{2} \ln(t^2+1) + \tan^{-1} t \right]_1^{\sqrt{3}}$$

$$= \left(4 \ln \sqrt{3} - \frac{1}{2} \ln 4 + \tan^{-1} \sqrt{3} \right) - \left(4 \ln 1 - \frac{1}{2} \ln 2 + \tan^{-1} 1 \right) = 2 \ln 3 - \ln 2 + \frac{\pi}{3} + \frac{1}{2} \ln 2 + \frac{\pi}{4}$$

$$= 2 \ln 3 - \frac{1}{2} \ln 2 + \frac{\pi}{12} = \ln \left(\frac{9}{\sqrt{2}} \right) + \frac{\pi}{12}$$

$$23. \frac{y^2+2y+1}{(y^2+1)^2} = \frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2+1)^2} \Rightarrow y^2+2y+1 = (Ay+B)(y^2+1) + Cy+D$$

$$= Ay^3 + By^2 + (A+C)y + (B+D) \Rightarrow A=0, B=1; A+C=2 \Rightarrow C=2; B+D=1 \Rightarrow D=0;$$

$$\int \frac{y^2+2y+1}{(y^2+1)^2} dy = \int \frac{1}{y^2+1} dy + 2 \int \frac{y}{(y^2+1)^2} dy = \tan^{-1} y - \frac{1}{y^2+1} + C$$

$$24. \frac{8x^2+8x+2}{(4x^2+1)^2} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{(4x^2+1)^2} \Rightarrow 8x^2+8x+2 = (Ax+B)(4x^2+1) + Cx+D$$

$$= 4Ax^3 + 4Bx^2 + (A+C)x + (B+D); A=0, B=2; A+C=8 \Rightarrow C=8; B+D=2 \Rightarrow D=0;$$

$$\int \frac{8x^2+8x+2}{(4x^2+1)^2} dx = 2 \int \frac{dx}{4x^2+1} + 8 \int \frac{x dx}{(4x^2+1)^2} = \tan^{-1} 2x - \frac{1}{4x^2+1} + C$$

$$25. \frac{2s+2}{(s^2+1)(s-1)^3} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{E}{(s-1)^3} \Rightarrow 2s+2$$

$$= (As+B)(s-1)^3 + C(s^2+1)(s-1)^2 + D(s^2+1)(s-1) + E(s^2+1)$$

$$= [As^4 + (-3A+B)s^3 + (3A-3B)s^2 + (-A+3B)s - B] + C(s^4 - 2s^3 + 2s^2 - 2s + 1) + D(s^3 - s^2 + s - 1) + E(s^2 + 1)$$

$$= (A+C)s^4 + (-3A+B-2C+D)s^3 + (3A-3B+2C-D+E)s^2 + (-A+3B-2C+D)s + (-B+C-D+E)$$

$$\Rightarrow \left. \begin{aligned} A+C &= 0 \\ -3A+B-2C+D &= 0 \\ 3A-3B+2C-D+E &= 0 \\ -A+3B-2C+D &= 2 \\ -B+C-D+E &= 2 \end{aligned} \right\} \text{summing all equations} \Rightarrow 2E=4 \Rightarrow E=2;$$

summing eqs (2) and (3) $\Rightarrow -2B + 2 = 0 \Rightarrow B = 1$; summing eqs (3) and (4) $\Rightarrow 2A + 2 = 2 \Rightarrow A = 0$; $C = 0$ from eq (1); then $-1 + 0 - D + 2 = 2$ from eq (5) $\Rightarrow D = -1$;

$$\int \frac{2s+2}{(s^2+1)(s-1)^3} ds = \int \frac{ds}{s^2+1} - \int \frac{ds}{(s-1)^2} + 2 \int \frac{ds}{(s-1)^3} = -(s-1)^{-2} + (s-1)^{-1} + \tan^{-1}s + C$$

$$\begin{aligned} 26. \frac{s^4+81}{s(s^2+9)^2} &= \frac{A}{s} + \frac{Bs+C}{s^2+9} + \frac{Ds+E}{(s^2+9)^2} \Rightarrow s^4+81 = A(s^2+9)^2 + (Bs+C)s(s^2+9) + (Ds+E)s \\ &= A(s^4+18s^2+81) + (Bs^4+Cs^3+9Bs^2+9Cs) + Ds^2+Es \\ &= (A+B)s^4 + Cs^3 + (18A+9B+D)s^2 + (9C+E)s + 81A \Rightarrow 81A = 81 \text{ or } A = 1; A+B = 1 \Rightarrow B = 0; \\ C &= 0; 9C+E = 0 \Rightarrow E = 0; 18A+9B+D = 0 \Rightarrow D = -18; \int \frac{s^4+81}{s(s^2+9)^2} ds = \int \frac{ds}{s} - 18 \int \frac{s ds}{(s^2+9)^2} \\ &= \ln|s| + \frac{9}{(s^2+9)} + C \end{aligned}$$

$$\begin{aligned} 27. \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2} &= \frac{A\theta+B}{\theta^2+2\theta+2} + \frac{C\theta+D}{(\theta^2+2\theta+2)^2} \Rightarrow 2\theta^3+5\theta^2+8\theta+4 = (A\theta+B)(\theta^2+2\theta+2) + C\theta+D \\ &= A\theta^3 + (2A+B)\theta^2 + (2A+2B+C)\theta + (2B+D) \Rightarrow A = 2; 2A+B = 5 \Rightarrow B = 1; 2A+2B+C = 8 \Rightarrow C = 2; \\ 2B+D &= 4 \Rightarrow D = 2; \int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2} d\theta = \int \frac{2\theta+1}{(\theta^2+2\theta+2)} d\theta + \int \frac{2\theta+2}{(\theta^2+2\theta+2)^2} d\theta \\ &= \int \frac{2\theta+2}{\theta^2+2\theta+2} d\theta - \int \frac{d\theta}{\theta^2+2\theta+2} + \int \frac{d(\theta^2+2\theta+2)}{(\theta^2+2\theta+2)^2} = \int \frac{d(\theta^2+2\theta+2)}{\theta^2+2\theta+2} - \int \frac{d\theta}{(\theta+1)^2+1} - \frac{1}{\theta^2+2\theta+2} \\ &= \frac{-1}{\theta^2+2\theta+2} + \ln(\theta^2+2\theta+2) - \tan^{-1}(\theta+1) + C \end{aligned}$$

$$\begin{aligned} 28. \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3} &= \frac{A\theta+B}{\theta^2+1} + \frac{C\theta+D}{(\theta^2+1)^2} + \frac{E\theta+F}{(\theta^2+1)^3} \Rightarrow \theta^4-4\theta^3+2\theta^2-3\theta+1 \\ &= (A\theta+B)(\theta^2+1)^2 + (C\theta+D)(\theta^2+1) + E\theta+F = (A\theta+B)(\theta^4+2\theta^2+1) + (C\theta^3+D\theta^2+C\theta+D) + E\theta+F \\ &= (A\theta^5+B\theta^4+2A\theta^3+2B\theta^2+A\theta+B) + (C\theta^3+D\theta^2+C\theta+D) + E\theta+F \\ &= A\theta^5+B\theta^4+(2A+C)\theta^3+(2B+D)\theta^2+(A+C+E)\theta+(B+D+F) \Rightarrow A = 0; B = 1; 2A+C = -4 \\ &\Rightarrow C = -4; 2B+D = 2 \Rightarrow D = 0; A+C+E = -3 \Rightarrow E = 1; B+D+F = 1 \Rightarrow F = 0; \\ \int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3} d\theta &= \int \frac{d\theta}{\theta^2+1} - 4 \int \frac{\theta d\theta}{(\theta^2+1)^2} + \int \frac{\theta d\theta}{(\theta^2+1)^3} = \tan^{-1}\theta + 2(\theta^2+1)^{-1} - \frac{1}{4}(\theta^2+1)^{-2} + C \end{aligned}$$

$$\begin{aligned} 29. \frac{2x^3-2x^2+1}{x^2-x} &= 2x + \frac{1}{x^2-x} = 2x + \frac{1}{x(x-1)}; \frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + Bx; x = 0 \Rightarrow A = -1; \\ x = 1 &\Rightarrow B = 1; \int \frac{2x^3-2x^2+1}{x^2-x} = \int 2x dx - \int \frac{dx}{x} + \int \frac{dx}{x-1} = x^2 - \ln|x| + \ln|x-1| + C = x^2 + \ln\left|\frac{x-1}{x}\right| + C \end{aligned}$$

$$30. \frac{x^4}{x^2-1} = (x^2+1) + \frac{1}{x^2-1} = (x^2+1) + \frac{1}{(x+1)(x-1)}; \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x+1);$$

$$x = -1 \Rightarrow A = -\frac{1}{2}; x = 1 \Rightarrow B = \frac{1}{2}; \int \frac{x^4}{x^2-1} dx = \int (x^2+1) dx - \frac{1}{2} \int \frac{dx}{x+1} + \frac{1}{2} \int \frac{dx}{x-1}$$

$$= \frac{1}{3}x^3 + x - \frac{1}{2} \ln|x+1| + \frac{1}{2} \ln|x-1| + C = \frac{x^3}{3} + x + \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C$$

$$31. \frac{9x^3-3x+1}{x^3-x^2} = 9 + \frac{9x^2-3x+1}{x^2(x-1)} \text{ (after long division); } \frac{9x^2-3x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$\Rightarrow 9x^2-3x+1 = Ax(x-1) + B(x-1) + Cx^2; x=1 \Rightarrow C=7; x=0 \Rightarrow B=-1; A+C=9 \Rightarrow A=2;$$

$$\int \frac{9x^3-3x+1}{x^3-x^2} dx = \int 9 dx + 2 \int \frac{dx}{x} - \int \frac{dx}{x^2} + 7 \int \frac{dx}{x-1} = 9x + 2 \ln|x| + \frac{1}{x} + 7 \ln|x-1| + C$$

$$32. \frac{16x^3}{4x^2-4x+1} = (4x+4) + \frac{12x-4}{4x^2-4x+1}; \frac{12x-4}{(2x-1)^2} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} \Rightarrow 12x-4 = A(2x-1) + B$$

$$\Rightarrow A=6; -A+B=-4 \Rightarrow B=2; \int \frac{16x^3}{4x^2-4x+1} dx = 4 \int (x+1) dx + 6 \int \frac{dx}{2x-1} + 2 \int \frac{dx}{(2x-1)^2}$$

$$= 2(x+1)^2 + 3 \ln|2x-1| - \frac{1}{2x-1} + C_1 = 2x^2 + 4x + 3 \ln|2x-1| - (2x-1)^{-1} + C, \text{ where } C = 2 + C_1$$

$$33. \frac{y^4+y^2-1}{y^3+y} = y - \frac{1}{y(y^2+1)}; \frac{1}{y(y^2+1)} = \frac{A}{y} + \frac{By+C}{y^2+1} \Rightarrow 1 = A(y^2+1) + (By+C)y = (A+B)y^2 + Cy + A$$

$$\Rightarrow A=1; A+B=0 \Rightarrow B=-1; C=0; \int \frac{y^4+y^2-1}{y^3+y} dy = \int y dy - \int \frac{dy}{y} + \int \frac{y dy}{y^2+1}$$

$$= \frac{y^2}{2} - \ln|y| + \frac{1}{2} \ln(1+y^2) + C$$

$$34. \frac{2y^4}{y^3-y^2+y-1} = 2y+2 + \frac{2}{y^3-y^2+y-1}; \frac{2}{y^3-y^2+y-1} = \frac{2}{(y^2+1)(y-1)} = \frac{A}{y-1} + \frac{By+C}{y^2+1}$$

$$\Rightarrow 2 = A(y^2+1) + (By+C)(y-1) = (Ay^2+A) + (By^2+Cy-By-C) = (A+B)y^2 + (-B+C)y + (A-C)$$

$$\Rightarrow A+B=0, -B+C=0 \text{ or } C=B, A-C=A-B=2 \Rightarrow A=1, B=-1, C=-1;$$

$$\int \frac{2y^4}{y^3-y^2+y-1} dy = 2 \int (y+1) dy + \int \frac{dy}{y-1} - \int \frac{y}{y^2+1} dy - \int \frac{dy}{y^2+1}$$

$$= (y+1)^2 + \ln|y-1| - \frac{1}{2} \ln(y^2+1) - \tan^{-1}y + C_1 = y^2 + 2y + \ln|y-1| - \frac{1}{2} \ln(y^2+1) - \tan^{-1}y + C,$$

where $C = C_1 + 1$

$$35. \int \frac{e^t dt}{e^{2t} + 3e^t + 2} = [e^t = y] \int \frac{dy}{y^2 + 3y + 2} = \int \frac{dy}{y+1} - \int \frac{dy}{y+2} = \ln \left| \frac{y+1}{y+2} \right| + C = \ln \left| \frac{e^t+1}{e^t+2} \right| + C$$

$$36. \int \frac{e^{4t} + 2e^{2t} - e^t}{e^{2t} + 1} dt; [e^t = y] \rightarrow \int \frac{y^3 + 2y - 1}{y^2 + 1} dy = \int \left(y + \frac{y-1}{y^2+1} \right) dy = \frac{y^2}{2} + \frac{1}{2} \ln(y^2+1) - \tan^{-1}y + C$$

$$= \frac{1}{2} e^{2t} - \tan^{-1}(e^t) + \frac{1}{2} \ln(e^{2t} + 1) + C$$

$$\begin{aligned} 37. \int \frac{\cos y \, dy}{\sin^2 y + \sin y - 6}; [\sin y = t, \cos y \, dy = dt] &\rightarrow \int \frac{dy}{t^2 + t - 6} = \frac{1}{5} \int \left(\frac{1}{t-2} - \frac{1}{t+3} \right) dt = \frac{1}{5} \ln \left| \frac{t-2}{t+3} \right| + C \\ &= \frac{1}{5} \ln \left| \frac{\sin y - 2}{\sin y + 3} \right| + C \end{aligned}$$

$$\begin{aligned} 38. \int \frac{\sin \theta \, d\theta}{\cos^2 \theta + \cos \theta - 2}; [\cos \theta = y] &\rightarrow - \int \frac{dy}{y^2 + y - 2} = -\frac{1}{3} \int \frac{dy}{y+2} - \frac{1}{3} \int \frac{dy}{y-1} = -\frac{1}{3} \ln \left| \frac{y+2}{y-1} \right| + C = \frac{1}{3} \ln \left| \frac{\cos \theta + 2}{\cos \theta - 1} \right| + C \\ &= \frac{1}{3} \ln \left| \frac{2 + \cos \theta}{1 - \cos \theta} \right| + C = -\frac{1}{3} \ln \left| \frac{\cos \theta - 1}{\cos \theta + 2} \right| + C \end{aligned}$$

$$\begin{aligned} 39. \int \frac{(x-2)^2 \tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2} dx &= \int \frac{\tan^{-1}(2x)}{4x^2+1} dx - 3 \int \frac{x}{(x-2)^2} dx \\ &= \frac{1}{2} \int \tan^{-1}(2x) d(\tan^{-1}(2x)) - 3 \int \frac{dx}{x-2} - 6 \int \frac{dx}{(x-2)^2} = \frac{(\tan^{-1} 2x)^2}{4} - 3 \ln |x-2| + \frac{6}{x-2} + C \end{aligned}$$

$$\begin{aligned} 40. \int \frac{(x+1)^2 \tan^{-1}(3x) + 9x^3 + x}{(9x^2+1)(x+1)^2} dx &= \int \frac{\tan^{-1}(3x)}{9x^2+1} dx + \int \frac{x}{(x+1)^2} dx \\ &= \frac{1}{3} \int \tan^{-1}(3x) d(\tan^{-1}(3x)) + \int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2} = \frac{(\tan^{-1} 3x)^2}{6} + \ln |x+1| + \frac{1}{x+1} + C \end{aligned}$$

$$\begin{aligned} 41. (t^2 - 3t + 2) \frac{dx}{dt} &= 1; x = \int \frac{dt}{t^2 - 3t + 2} = \int \frac{dt}{t-2} - \int \frac{dt}{t-1} = \ln \left| \frac{t-2}{t-1} \right| + C; \frac{t-2}{t-1} = Ce^x; t = 3 \text{ and } x = 0 \\ &\Rightarrow \frac{1}{2} = C \Rightarrow \frac{t-2}{t-1} = \frac{1}{2} e^x \Rightarrow x = \ln \left| 2 \left(\frac{t-2}{t-1} \right) \right| = \ln |t-2| - \ln |t-1| + \ln 2 \end{aligned}$$

$$\begin{aligned} 42. (3t^4 + 4t^2 + 1) \frac{dx}{dt} &= 2\sqrt{3}; x = 2\sqrt{3} \int \frac{dt}{3t^4 + 4t^2 + 1} = \sqrt{3} \int \frac{dt}{t^2 + \frac{1}{3}} - \sqrt{3} \int \frac{dt}{t^2 + 1} \\ &= 3 \tan^{-1}(\sqrt{3}t) - \sqrt{3} \tan^{-1} t + C; t = 1 \text{ and } x = \frac{-\pi\sqrt{3}}{4} \Rightarrow -\frac{\sqrt{3}\pi}{4} = \pi - \frac{\sqrt{3}}{4}\pi + C \Rightarrow C = -\pi \\ &\Rightarrow x = 3 \tan^{-1}(\sqrt{3}t) - \sqrt{3} \tan^{-1} t - \pi \end{aligned}$$

$$\begin{aligned} 43. (t^2 + 2t) \frac{dx}{dt} &= 2x + 2; \frac{1}{2} \int \frac{dx}{x+1} = \int \frac{dt}{t^2 + 2t} \Rightarrow \frac{1}{2} \ln |x+1| = \frac{1}{2} \int \frac{dt}{t} - \frac{1}{2} \int \frac{dt}{t+2} \Rightarrow \ln |x+1| = \ln \left| \frac{t}{t+2} \right| + C; \\ t = 1 \text{ and } x = 1 &\Rightarrow \ln 2 = \ln \frac{1}{3} + C \Rightarrow C = \ln 2 + \ln 3 = \ln 6 \Rightarrow \ln |x+1| = \ln 6 \left| \frac{t}{t+2} \right| \Rightarrow x+1 = \frac{6t}{t+2} \\ &\Rightarrow x = \frac{6t}{t+2} - 1, t > 0 \end{aligned}$$

$$44. (t+1) \frac{dx}{dt} = x^2 + 1 \Rightarrow \int \frac{dx}{x^2 + 1} = \int \frac{dt}{t+1} \Rightarrow \tan^{-1} x = \ln |t+1| + C; t=0 \text{ and } x = \frac{\pi}{4} \Rightarrow \tan^{-1} \frac{\pi}{4} = \ln |1| + C \\ \Rightarrow C = \tan^{-1} \frac{\pi}{4} = 1 \Rightarrow \tan^{-1} x = \ln |t+1| + 1 \Rightarrow x = \tan(\ln(t+1) + 1), t > -1$$

$$45. \frac{1}{y^2 - y} dy = e^x dx \Rightarrow \int \frac{1}{y(y-1)} dy = \int e^x dx = e^x + C$$

$$\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1} \Rightarrow 1 = A(y-1) + B(y) = (A+B)y - A$$

Equating coefficients of like terms gives

$$A + B = 0 \text{ and } -A = 1$$

Solving the system simultaneously yields $A = -1$, $B = 1$.

$$\int \frac{1}{y(y-1)} dy = \int -\frac{1}{y} dy + \int \frac{1}{y-1} dy = -\ln|y| + \ln|y-1| + C_2 \Rightarrow -\ln|y| + \ln|y-1| = e^x + C$$

Substitute $x = 0$, $y = 2$.

$$-\ln 2 + 0 = 1 + C \text{ or } C = -1 - \ln 2$$

The solution to the initial value problem is

$$-\ln|y| + \ln|y-1| = e^x - 1 - \ln 2.$$

$$46. \frac{1}{(y+1)^2} dy = \sin \theta d\theta \Rightarrow \int \frac{1}{(y+1)^2} dy = \int \sin \theta d\theta \Rightarrow -\frac{1}{y+1} = -\cos \theta + C$$

$$\text{Substitute } \theta = \frac{\pi}{2}, y = 0 \Rightarrow -1 = 0 + C \text{ or } C = -1$$

The solution to the initial value problem is

$$-\frac{1}{y+1} = -\cos \theta - 1 \Rightarrow y+1 = \frac{1}{\cos \theta + 1} \Rightarrow y = \frac{1}{\cos \theta + 1} - 1$$

$$47. dy = \frac{dx}{x^2 - 3x + 2}; x^2 - 3x + 2 = (x-2)(x-1) \Rightarrow \frac{1}{x^2 - 3x + 2} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x-2)$$

$$\Rightarrow 1 = (A+B)x - A - 2B$$

Equating coefficients of like terms gives

$$A + B = 0, -A - 2B = 1$$

Solving the system simultaneously yields $A = 1$, $B = -1$.

$$\int dy = \int \frac{dx}{x^2 - 3x + 2} = \int \frac{dx}{x-2} - \int \frac{dx}{x-1}$$

$$y = \ln|x-2| - \ln|x-1| + C$$

$$\text{Substitute } x = 3, y = 0 \Rightarrow 0 = 0 - \ln 2 + C \text{ or } C = \ln 2$$

The solution to the initial value problems is

$$y = \ln|x-2| - \ln|x-1| + \ln 2$$

$$48. \frac{ds}{2s+2} = \frac{dt}{t^2+2t} \Rightarrow \int \frac{ds}{2s+2} = \frac{1}{2} \int \frac{ds}{s+1} = \frac{1}{2} \ln|s+1| + C_1$$

$$t^2 + 2t = t(t+2) \Rightarrow \frac{1}{t^2+2t} = \frac{A}{t} + \frac{B}{t+2} \Rightarrow 1 = A(t+2) + Bt \Rightarrow 1 = (A+B)t + 2A$$

Equating coefficients of like terms gives $A+B=0$ and $2A=1$

Solving the system simultaneously yields $A = \frac{1}{2}$, $B = -\frac{1}{2}$.

$$\int \frac{dt}{t^2+2t} = \int \frac{1/2}{t} dt - \int \frac{1/2}{t+2} dt = \frac{1}{2} \ln|t| - \frac{1}{2} \ln|t+2| + C_2 \Rightarrow \frac{1}{2} \ln|s+1| = \frac{1}{2} \ln|t| - \frac{1}{2} \ln|t+2| + C_3$$

$$\Rightarrow \ln|s+1| = \ln|t| - \ln|t+2| + C$$

Substitute $t=1$, $s=1 \Rightarrow \ln 2 = 0 - \ln 3 + C$ or $C = \ln 2 + \ln 3 = \ln 6$

The solution to the initial value problem is

$$\ln|s+1| = \ln|t| - \ln|t+2| + \ln 6 \Rightarrow \ln|s+1| = \ln \left| \frac{6t}{t+2} \right| \Rightarrow |s+1| = \left| \frac{6t}{t+2} \right|.$$

$$49. V = \pi \int_{0.5}^{2.5} y^2 dx = \pi \int_{0.5}^{2.5} \frac{9}{3x-x^2} dx = 3\pi \left(\int_{0.5}^{2.5} \left(-\frac{1}{x-3} + \frac{1}{x} \right) dx \right) = \left[3\pi \ln \left| \frac{x}{x-3} \right| \right]_{0.5}^{2.5} = 3\pi \ln 25$$

$$50. V = 2\pi \int_0^1 xy dx = 2\pi \int_0^1 \frac{2x}{(x+1)(2-x)} dx = 4\pi \int_0^1 \left(-\frac{1}{3} \left(\frac{1}{x+1} \right) + \frac{2}{3} \left(\frac{1}{2-x} \right) \right) dx$$

$$= \left[-\frac{4\pi}{3} (\ln|x+1| + 2 \ln|2-x|) \right]_0^1 = \frac{4\pi}{3} (\ln 2)$$

$$51. (a) \frac{dx}{dt} = kx(N-x) \Rightarrow \int \frac{dx}{x(N-x)} = \int k dt \Rightarrow \frac{1}{N} \int \frac{dx}{x} + \frac{1}{N} \int \frac{dx}{N-x} = \int k dt \Rightarrow \frac{1}{N} \ln \left| \frac{x}{N-x} \right| = kt + C;$$

$$k = \frac{1}{250}, N = 1000, t = 0 \text{ and } x = 2 \Rightarrow \frac{1}{1000} \ln \left| \frac{2}{998} \right| = C \Rightarrow \frac{1}{1000} \ln \left| \frac{x}{1000-x} \right| = \frac{t}{250} + \frac{1}{1000} \ln \left(\frac{1}{499} \right)$$

$$\Rightarrow \ln \left| \frac{499x}{1000-x} \right| = 4t \Rightarrow \frac{499x}{1000-x} = e^{4t} \Rightarrow 499x = e^{4t}(1000-x) \Rightarrow (499 + e^{4t})x = 1000e^{4t} \Rightarrow x = \frac{1000e^{4t}}{499 + e^{4t}}$$

$$(b) x = \frac{1}{2}N = 500 \Rightarrow 500 = \frac{1000e^{4t}}{499 + e^{4t}} \Rightarrow 500 \cdot 499 + 500e^{4t} = 1000e^{4t} \Rightarrow e^{4t} = 499 \Rightarrow t = \frac{1}{4} \ln 499 \approx 1.55 \text{ days}$$

$$52. \frac{dx}{dt} = k(a-x)(b-x) \Rightarrow \frac{dx}{(a-x)(b-x)} = k dt$$

$$(a) a = b: \int \frac{dx}{(a-x)^2} = \int k dt \Rightarrow \frac{1}{a-x} = kt + C; t = 0 \text{ and } x = 0 \Rightarrow \frac{1}{a} = C \Rightarrow \frac{1}{a-x} = kt + \frac{1}{a}$$

$$\Rightarrow \frac{1}{a-x} = \frac{akt+1}{a} \Rightarrow a-x = \frac{a}{akt+1} \Rightarrow x = a - \frac{a}{akt+1} = \frac{a^2kt}{akt+1}$$

$$(b) a \neq b: \int \frac{dx}{(a-x)(b-x)} = \int k dt \Rightarrow \frac{1}{b-a} \int \frac{dx}{a-x} - \frac{1}{b-a} \int \frac{dx}{b-x} = \int k dt \Rightarrow \frac{1}{b-a} \ln \left| \frac{b-x}{a-x} \right| = kt + C;$$

$$t = 0 \text{ and } x = 0 \Rightarrow \frac{1}{b-a} \ln \frac{b}{a} = C \Rightarrow \ln \left| \frac{b-x}{a-x} \right| = (b-a)kt + \ln \left(\frac{b}{a} \right) \Rightarrow \frac{b-x}{a-x} = \frac{b}{a} e^{(b-a)kt}$$

$$\Rightarrow x = \frac{ab[1 - e^{(a-b)kt}]}{a - be^{(a-b)kt}}$$

7.4 TRIGONOMETRIC SUBSTITUTIONS

$$1. y = 3 \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dy = \frac{3 d\theta}{\cos^2 \theta}, 9 + y^2 = 9(1 + \tan^2 \theta) = \frac{9}{\cos^2 \theta} \Rightarrow \frac{1}{\sqrt{9 + y^2}} = \frac{|\cos \theta|}{3} = \frac{\cos \theta}{3}$$

(because $\cos \theta > 0$ when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$);

$$\int \frac{dy}{\sqrt{9 + y^2}} = 3 \int \frac{\cos \theta d\theta}{3 \cos^2 \theta} = \int \frac{d\theta}{\cos \theta} = \ln |\sec \theta + \tan \theta| + C' = \ln \left| \frac{\sqrt{9 + y^2}}{3} + \frac{y}{3} \right| + C' = \ln |\sqrt{9 + y^2} + y| + C$$

$$2. \int \frac{3 dy}{\sqrt{1 + 9y^2}}; [3y = x] \rightarrow \int \frac{dx}{\sqrt{1 + x^2}}; x = \tan t, -\frac{\pi}{2} < t < \frac{\pi}{2}, dx = \frac{dt}{\cos^2 t}, \sqrt{1 + x^2} = \frac{1}{\cos t};$$

$$\int \frac{dx}{\sqrt{1 + x^2}} = \int \frac{dt}{\cos^2 t \left(\frac{1}{\cos t} \right)} = \ln |\sec t + \tan t| + C = \ln |\sqrt{x^2 + 1} + x| + C = \ln |\sqrt{1 + 9y^2} + 3y| + C$$

$$3. t = 5 \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dt = 5 \cos \theta d\theta, \sqrt{25 - t^2} = 5 \cos \theta;$$

$$\begin{aligned} \int \sqrt{25 - t^2} dt &= \int (5 \cos \theta)(5 \cos \theta) d\theta = 25 \int \cos^2 \theta d\theta = 25 \int \frac{1 + \cos 2\theta}{2} d\theta = 25 \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) + C \\ &= \frac{25}{2}(\theta + \sin \theta \cos \theta) + C = \frac{25}{2} \left[\sin^{-1} \left(\frac{t}{5} \right) + \left(\frac{t}{5} \right) \left(\frac{\sqrt{25 - t^2}}{5} \right) \right] + C = \frac{25}{2} \sin^{-1} \left(\frac{t}{5} \right) + \frac{t\sqrt{25 - t^2}}{2} + C \end{aligned}$$

$$4. t = \frac{1}{3} \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dt = \frac{1}{3} \cos \theta d\theta, \sqrt{1 - 9t^2} = \cos \theta;$$

$$\int \sqrt{1 - 9t^2} dt = \frac{1}{3} \int (\cos \theta)(\cos \theta) d\theta = \frac{1}{3} \int \cos^2 \theta d\theta = \frac{1}{6}(\theta + \sin \theta \cos \theta) + C = \frac{1}{6} \left[\sin^{-1}(3t) + 3t\sqrt{1 - 9t^2} \right] + C$$

$$5. x = \frac{7}{2} \sec \theta, 0 < \theta < \frac{\pi}{2}, dx = \frac{7}{2} \sec \theta \tan \theta d\theta, \sqrt{4x^2 - 49} = \sqrt{49 \sec^2 \theta - 49} = 7 \tan \theta;$$

$$\int \frac{dx}{\sqrt{4x^2 - 49}} = \int \frac{\left(\frac{7}{2} \sec \theta \tan \theta \right) d\theta}{7 \tan \theta} = \frac{1}{2} \int \sec \theta d\theta = \frac{1}{2} \ln |\sec \theta + \tan \theta| + C = \frac{1}{2} \ln \left| \frac{2x}{7} + \frac{\sqrt{4x^2 - 49}}{7} \right| + C$$

$$6. x = \frac{3}{5} \sec \theta, 0 < \theta < \frac{\pi}{2}, dx = \frac{3}{5} \sec \theta \tan \theta d\theta, \sqrt{25x^2 - 9} = \sqrt{9 \sec^2 \theta - 9} = 3 \tan \theta;$$

$$\int \frac{5 dx}{\sqrt{25x^2 - 9}} = \int \frac{5 \left(\frac{3}{5} \sec \theta \tan \theta \right) d\theta}{3 \tan \theta} = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C = \ln \left| \frac{5x}{3} + \frac{\sqrt{25x^2 - 9}}{3} \right| + C$$

7. $x = \sec \theta$, $0 < \theta < \frac{\pi}{2}$, $dx = \sec \theta \tan \theta d\theta$, $\sqrt{x^2 - 1} = \tan \theta$;

$$\int \frac{dx}{x^2 \sqrt{x^2 - 1}} = \int \frac{\sec \theta \tan \theta d\theta}{\sec^2 \theta \tan \theta} = \int \frac{d\theta}{\sec \theta} = \sin \theta + C = \tan \theta \cos \theta + C = \frac{\sqrt{x^2 - 1}}{x} + C$$

8. $x = \sec \theta$, $0 < \theta < \frac{\pi}{2}$, $dx = \sec \theta \tan \theta d\theta$, $\sqrt{x^2 - 1} = \tan \theta$;

$$\begin{aligned} \int \frac{2 dx}{x^3 \sqrt{x^2 - 1}} &= \int \frac{2 \tan \theta \sec \theta d\theta}{\sec^3 \theta \tan \theta} = 2 \int \cos^2 \theta d\theta = 2 \int \left(\frac{1 + \cos 2\theta}{2} \right) d\theta = \theta + \sin \theta \cos \theta + C \\ &= \theta + \tan \theta \cos^2 \theta + C = \sec^{-1} x + \sqrt{x^2 - 1} \left(\frac{1}{x} \right)^2 + C = \sec^{-1} x + \frac{\sqrt{x^2 - 1}}{x^2} + C \end{aligned}$$

9. $x = 2 \tan \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $dx = \frac{2 d\theta}{\cos^2 \theta}$, $\sqrt{x^2 + 4} = \frac{2}{\cos \theta}$;

$$\int \frac{x^3 dx}{\sqrt{x^2 + 4}} = \int \frac{(8 \tan^3 \theta)(\cos \theta) d\theta}{\cos^2 \theta} = 8 \int \frac{\sin^3 \theta d\theta}{\cos^4 \theta} = 8 \int \frac{(\cos^2 \theta - 1)(-\sin \theta) d\theta}{\cos^4 \theta};$$

$$\begin{aligned} [t = \cos \theta] \rightarrow 8 \int \frac{t^2 - 1}{t^4} dt &= 8 \int \left(\frac{1}{t^2} - \frac{1}{t^4} \right) dt = 8 \left(-\frac{1}{t} + \frac{1}{3t^3} \right) + C = 8 \left(-\sec \theta + \frac{\sec^3 \theta}{3} \right) + C \\ &= 8 \left(-\frac{\sqrt{x^2 + 4}}{2} + \frac{(x^2 + 4)^{3/2}}{8 \cdot 3} \right) + C = \frac{1}{3}(x^2 + 4)^{3/2} - 4\sqrt{x^2 + 4} + C \end{aligned}$$

10. $x = \tan \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $dx = \sec^2 \theta d\theta$, $\sqrt{x^2 + 1} = \sec \theta$;

$$\int \frac{dx}{x^2 \sqrt{x^2 + 1}} = \int \frac{\sec^2 \theta d\theta}{\tan^2 \theta \sec \theta} = \int \frac{\cos \theta d\theta}{\sin^2 \theta} = -\frac{1}{\sin \theta} + C = \frac{-\sqrt{x^2 + 1}}{x} + C$$

11. $w = 2 \sin \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $dw = 2 \cos \theta d\theta$, $\sqrt{4 - w^2} = 2 \cos \theta$;

$$\int \frac{8 dw}{w^2 \sqrt{4 - w^2}} = \int \frac{8 \cdot 2 \cos \theta d\theta}{4 \sin^2 \theta \cdot 2 \cos \theta} = 2 \int \frac{d\theta}{\sin^2 \theta} = -2 \cot \theta + C = \frac{-2\sqrt{4 - w^2}}{w} + C$$

12. $w = 3 \sin \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $dw = 3 \cos \theta d\theta$, $\sqrt{9 - w^2} = 3 \cos \theta$;

$$\begin{aligned} \int \frac{\sqrt{9 - w^2}}{w^2} dw &= \int \frac{3 \cos \theta \cdot 3 \cos \theta d\theta}{9 \sin^2 \theta} = \int \cot^2 \theta d\theta = \int \left(\frac{1 - \sin^2 \theta}{\sin^2 \theta} \right) d\theta = \int (\csc^2 \theta - 1) d\theta \\ &= -\cot \theta - \theta + C = -\frac{\sqrt{9 - w^2}}{w} - \sin^{-1} \left(\frac{w}{3} \right) + C \end{aligned}$$

13. $x = \sec \theta$, $0 < \theta < \frac{\pi}{2}$, $dx = \sec \theta \tan \theta d\theta$, $(x^2 - 1)^{3/2} = \tan^3 \theta$;

$$\begin{aligned} \int \frac{dx}{(x^2 - 1)^{3/2}} &= \int \frac{\sec \theta \tan \theta d\theta}{\tan^3 \theta} = \int \frac{\cos \theta d\theta}{\sin^2 \theta} = -\frac{1}{\sin \theta} + C = -\left(\frac{1}{\tan \theta} \right) \left(\frac{1}{\cos \theta} \right) + C \\ &= -\left(\frac{1}{\sqrt{x^2 - 1}} \right) (x) + C = -\frac{x}{\sqrt{x^2 - 1}} + C \end{aligned}$$

14. $x = \sec \theta$, $0 < \theta < \frac{\pi}{2}$, $dx = \sec \theta \tan \theta d\theta$, $(x^2 - 1)^{5/2} = \tan^5 \theta$;

$$\begin{aligned} \int \frac{x^2 dx}{(x^2 - 1)^{5/2}} &= \int \frac{\sec^2 \theta \cdot \sec \theta \tan \theta d\theta}{\tan^5 \theta} = \int \frac{\cos \theta}{\sin^4 \theta} d\theta = -\frac{1}{3 \sin^3 \theta} + C = -\frac{1}{3} \left(\frac{1}{\tan^3 \theta} \right) \left(\frac{1}{\cos^3 \theta} \right) + C \\ &= -\frac{1}{3} \left[\frac{1}{(x^2 - 1)^{3/2}} \right] (x^3) + C = -\frac{x^3}{3(x^2 - 1)^{3/2}} + C \end{aligned}$$

15. $x = \sin \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $dx = \cos \theta d\theta$, $(1 - x^2)^{3/2} = \cos^3 \theta$;

$$\int \frac{(1 - x^2)^{3/2} dx}{x^6} = \int \frac{\cos^3 \theta \cdot \cos \theta d\theta}{\sin^6 \theta} = \int \cot^4 \theta \csc^2 \theta d\theta = -\frac{\cot^5 \theta}{5} + C = -\frac{1}{5} \left(\frac{\sqrt{1 - x^2}}{x} \right)^5 + C$$

16. $x = \sin \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $dx = \cos \theta d\theta$, $(1 - x^2)^{1/2} = \cos \theta$;

$$\int \frac{(1 - x^2)^{1/2} dx}{x^4} = \int \frac{\cos \theta \cdot \cos \theta d\theta}{\sin^4 \theta} = \int \cot^2 \theta \csc^2 \theta d\theta = -\frac{\cot^3 \theta}{3} + C = -\frac{1}{3} \left(\frac{\sqrt{1 - x^2}}{x} \right)^3 + C$$

17. $x = \frac{1}{2} \tan \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $dx = \frac{1}{2} \sec^2 \theta d\theta$, $(4x^2 + 1)^2 = \sec^4 \theta$;

$$\begin{aligned} \int \frac{8 dx}{(4x^2 + 1)^2} &= \int \frac{8 \left(\frac{1}{2} \sec^2 \theta \right) d\theta}{\sec^4 \theta} = 4 \int \cos^2 \theta d\theta = 2(\theta + \sin \theta \cos \theta) + C = 2(\theta + \tan \theta + \cos^2 \theta) + C \\ &= 2 \tan^{-1} 2x + \frac{4x}{(4x^2 + 1)} + C \end{aligned}$$

18. $t = \frac{1}{3} \tan \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $dt = \frac{1}{3} \sec^2 \theta d\theta$, $9t^2 + 1 = \sec^2 \theta$;

$$\begin{aligned} \int \frac{6 dt}{(9t^2 + 1)^2} &= \int \frac{6 \left(\frac{1}{3} \sec^2 \theta \right) d\theta}{\sec^4 \theta} = 2 \int \cos^2 \theta d\theta = \theta + \sin \theta \cos \theta + C = \theta + \tan \theta \cos^2 \theta + C \\ &= \tan^{-1} 3t + \frac{3t}{(9t^2 + 1)} + C \end{aligned}$$

19. Let $e^t = 3 \tan \theta$, $t = \ln(3 \tan \theta)$, $dt = \frac{\sec^2 \theta}{\tan \theta} d\theta$, $\sqrt{e^{2t} + 9} = \sqrt{9 \tan^2 \theta + 9} = 3 \sec \theta$;

$$\begin{aligned} \int_0^{\ln 4} \frac{e^t dt}{\sqrt{e^{2t} + 9}} &= \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \frac{3 \tan \theta \cdot \sec^2 \theta d\theta}{\tan \theta \cdot 3 \sec \theta} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \sec \theta d\theta = [\ln |\sec \theta + \tan \theta|]_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \\ &= \ln \left(\frac{5}{3} + \frac{4}{3} \right) - \ln \left(\frac{\sqrt{10}}{3} + \frac{1}{3} \right) = \ln 9 - \ln(1 + \sqrt{10}) \end{aligned}$$

20. Let $e^t = \tan \theta$, $t = \ln(\tan \theta)$, $\frac{3}{4} \leq \theta \leq \frac{4}{3}$, $dt = \frac{\sec^2 \theta}{\tan^2 \theta} d\theta$, $1 + e^{2t} = 1 + \tan^2 \theta = \sec^2 \theta$;

$$\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t dt}{(1 + e^{2t})^{3/2}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{(\tan \theta) \left(\frac{\sec^2 \theta}{\tan^2 \theta} \right) d\theta}{\sec^3 \theta} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \cos \theta d\theta = [\sin \theta]_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$$

21. $\int_{1/12}^{1/4} \frac{2 dt}{\sqrt{t + 4t\sqrt{t}}}$; $[u = 2\sqrt{t}, du = \frac{1}{\sqrt{t}} dt] \rightarrow \int_{1/\sqrt{3}}^1 \frac{2 du}{1 + u^2}$; $u = \tan \theta$, $\frac{\pi}{6} < \theta < \frac{\pi}{4}$, $du = \sec^2 \theta d\theta$, $1 + u^2 = \sec^2 \theta$;

$$\int_{1/\sqrt{3}}^1 \frac{4 du}{u(1 + u^2)} = \int_{\pi/6}^{\pi/4} \frac{2 \sec^2 \theta d\theta}{\sec^2 \theta} = [2\theta]_{\pi/6}^{\pi/4} = 2\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\pi}{6}$$

22. $y = e^{\tan \theta}$, $dy = e^{\tan \theta} \sec^2 \theta d\theta$, $\sqrt{1 + (\ln y)^2} = \sqrt{1 + \tan^2 \theta} = \sec \theta$;

$$\int_1^e \frac{dy}{y\sqrt{1 + (\ln y)^2}} = \int_0^{\pi/4} \frac{e^{\tan \theta} \sec^2 \theta}{e^{\tan \theta} \sec \theta} d\theta = \int_0^{\pi/4} \sec \theta d\theta = [\ln |\sec \theta + \tan \theta|]_0^{\pi/4} = \ln(1 + \sqrt{2})$$

23. $x = \sec \theta$, $dx = \sec \theta \tan \theta d\theta$, $\sqrt{x^2 - 1} = \sqrt{\sec^2 \theta - 1} = \tan \theta$;

$$\int \frac{dx}{x\sqrt{x^2 - 1}} = \int \frac{\sec \theta \tan \theta d\theta}{\sec \theta \tan \theta} = \theta + C = \sec^{-1} |x| + C$$

24. $x = \tan \theta$, $dx = \sec^2 \theta d\theta$, $1 + x^2 = \sec^2 \theta$;

$$\int \frac{dx}{x^2 + 1} = \int \frac{\sec^2 \theta d\theta}{\sec^2 \theta} = \theta + C = \tan^{-1} x + C$$

25. $x = \sec \theta$, $dx = \sec \theta \tan \theta d\theta$, $\sqrt{x^2 - 1} = \sqrt{\sec^2 \theta - 1} = \tan \theta$;

$$\int \frac{x dx}{\sqrt{x^2 - 1}} = \int \frac{\sec \theta \cdot \sec \theta \tan \theta d\theta}{\tan \theta} = \int \sec^2 \theta d\theta = \tan \theta + C = \sqrt{x^2 - 1} + C$$

26. $x = \sin \theta$, $dx = \cos \theta d\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$;

$$\int \frac{dx}{\sqrt{1 - x^2}} = \int \frac{\cos \theta d\theta}{\cos \theta} = \theta + C = \sin^{-1} x + C$$

27. $x \frac{dy}{dx} = \sqrt{x^2 - 4}$; $dy = \sqrt{x^2 - 4} \frac{dx}{x}$; $y = \int \frac{\sqrt{x^2 - 4}}{x} dx$; $\left[\begin{array}{l} x = 2 \sec \theta, 0 < \theta < \frac{\pi}{2} \\ dx = 2 \sec \theta \tan \theta d\theta \\ \sqrt{x^2 - 4} = 2 \tan \theta \end{array} \right]$

$$\rightarrow y = \int \frac{(2 \tan \theta)(2 \sec \theta \tan \theta) d\theta}{2 \sec \theta} = 2 \int \tan^2 \theta d\theta = 2 \int (\sec^2 \theta - 1) d\theta = 2(\tan \theta - \theta) + C$$

$$= 2 \left[\frac{\sqrt{x^2 - 4}}{2} - \sec^{-1} \left(\frac{x}{2} \right) \right] + C; x = 2 \text{ and } y = 0 \Rightarrow 0 = 0 + C \Rightarrow C = 0 \Rightarrow y = 2 \left[\frac{\sqrt{x^2 - 4}}{2} - \sec^{-1} \frac{x}{2} \right]$$

$$\begin{aligned} 28. \sqrt{x^2 - 9} \frac{dy}{dx} &= 1, dy = \frac{dx}{\sqrt{x^2 - 9}}; y = \int \frac{dx}{\sqrt{x^2 - 9}}; \left[\begin{array}{l} x = 3 \sec \theta, 0 < \theta < \frac{\pi}{2} \\ dx = 3 \sec \theta \tan \theta d\theta \\ \sqrt{x^2 - 9} = 3 \tan \theta \end{array} \right] \rightarrow y = \int \frac{3 \sec \theta \tan \theta d\theta}{3 \tan \theta} \\ &= \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C = \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| + C; x = 5 \text{ and } y = \ln 3 \Rightarrow \ln 3 = \ln 3 + C \Rightarrow C = 0 \\ &\Rightarrow y = \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| \end{aligned}$$

$$\begin{aligned} 29. (x^2 + 4) \frac{dy}{dx} &= 3, dy = \frac{3 dx}{x^2 + 4}; y = 3 \int \frac{dx}{x^2 + 4} = \frac{3}{2} \tan^{-1} \frac{x}{2} + C; x = 2 \text{ and } y = 0 \Rightarrow 0 = \frac{3}{2} \tan^{-1} 1 + C \\ &\Rightarrow C = -\frac{3\pi}{8} \Rightarrow y = \frac{3}{2} \tan^{-1} \left(\frac{x}{2} \right) - \frac{3\pi}{8} \end{aligned}$$

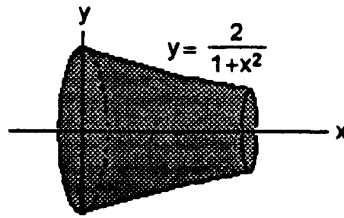
$$\begin{aligned} 30. (x^2 + 1)^2 \frac{dy}{dx} &= \sqrt{x^2 + 1}, dy = \frac{dx}{(x^2 + 1)^{3/2}}; x = \tan \theta, dx = \sec^2 \theta d\theta, (x^2 + 1)^{3/2} = \sec^3 \theta; \\ y &= \int \frac{\sec^2 \theta d\theta}{\sec^3 \theta} = \int \cos \theta d\theta = \sin \theta + C = \tan \theta \cos \theta + C = \frac{\tan \theta}{\sec \theta} + C = \frac{x}{\sqrt{x^2 + 1}} + C; x = 0 \text{ and } y = 1 \\ &\Rightarrow 1 = 0 + C \Rightarrow y = \frac{x}{\sqrt{x^2 + 1}} + 1 \end{aligned}$$

$$\begin{aligned} 31. A &= \int_0^3 \frac{\sqrt{9 - x^2}}{3} dx; x = 3 \sin \theta, 0 \leq \theta \leq \frac{\pi}{2}, dx = 3 \cos \theta d\theta, \sqrt{9 - x^2} = \sqrt{9 - 9 \sin^2 \theta} = 3 \cos \theta; \\ A &= \int_0^{\pi/2} \frac{3 \cos \theta \cdot 3 \cos \theta d\theta}{3} = 3 \int_0^{\pi/2} \cos^2 \theta d\theta = \frac{3}{2} [\theta + \sin \theta \cos \theta]_0^{\pi/2} = \frac{3\pi}{4} \end{aligned}$$

$$32. V = \int_0^1 \pi \left(\frac{2}{1 + x^2} \right)^2 dx = 4\pi \int_0^1 \frac{dx}{(x^2 + 1)^2};$$

$$x = \tan \theta, dx = \sec^2 \theta d\theta, x^2 + 1 = \sec^2 \theta;$$

$$V = 4\pi \int_0^{\pi/4} \frac{\sec^2 \theta d\theta}{\sec^4 \theta} = 4\pi \int_0^{\pi/4} \cos^2 \theta d\theta = 2\pi \int_0^{\pi/4} (1 + \cos 2\theta) d\theta = 2\pi \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/4} = \pi \left(\frac{\pi}{2} + 1 \right)$$



$$33. (a) \text{ From the figure, } \tan \frac{x}{2} = \frac{\sin x}{1 + \cos x}$$

$$(b) \text{ From part (a), } z = \frac{\sin x}{1 + \cos x} \Rightarrow z(1 + \cos x) = \sin x \Rightarrow z^2(1 + \cos x)^2 = \sin^2 x$$

$$\Rightarrow z^2(1 + \cos x)^2 - (1 - \cos x)(1 + \cos x) = 0 \Rightarrow (1 + \cos x)(z^2 + z^2 \cos x - 1 + \cos x) = 0$$

$$1 + \cos x = 0 \quad \text{or} \quad (z^2 + 1) \cos x = 1 - z^2$$

$$\cos x = -1 \quad \cos x = \frac{1 - z^2}{1 + z^2}$$

$\cos x = -1$ does not make sense in this case.

$$(c) \text{ From part (b), } \cos x = \frac{1 - z^2}{1 + z^2} \Rightarrow \sin^2 x = 1 - \frac{(1 - z^2)^2}{(1 + z^2)^2} = \frac{(1 + z^2)^2 - (1 - z^2)^2}{(1 + z^2)^2}$$

$$= \frac{1 + 2z^2 + z^4 - 1 + 2z^2 - z^4}{(1 + z^2)^2} = \frac{4z^2}{(1 + z^2)^2} \Rightarrow \sin x = \pm \frac{2z}{1 + z^2}$$

Only $\sin x = \frac{2z}{1 + z^2}$ makes sense in this case.

$$(d) \quad z = \tan \frac{x}{2}, \quad dz = \left(\frac{1}{2} \sec^2 \frac{x}{2} \right) dx \Rightarrow dz = \frac{1}{2} (1 + \tan^2 \frac{x}{2}) dx \Rightarrow dz = \frac{1}{2} (1 + z^2) dx \Rightarrow dx = \frac{2 dz}{1 + z^2}$$

$$34. \quad \int \frac{dx}{1 + \sin x} = \int \frac{\frac{2 dz}{1 + z^2}}{1 + \frac{2z}{1 + z^2}} = \int \frac{2 dz}{z^2 + 2z + 1} = \int \frac{2 dz}{(z + 1)^2} = -\frac{2}{z + 1} + C = -\frac{2}{\tan \frac{x}{2} + 1} + C$$

$$35. \quad \int \frac{dx}{1 - \cos x} = \int \frac{\frac{2 dz}{1 + z^2}}{1 - \frac{2z}{1 + z^2}} = \int \frac{dz}{z^2} = -\frac{1}{z} + C = -\frac{1}{\tan \frac{x}{2}} + C$$

$$36. \quad \int \frac{d\theta}{1 - \sin \theta} = \int \frac{\frac{2 dz}{1 + z^2}}{1 - \frac{2z}{1 + z^2}} = \int \frac{2 dz}{z^2 - 2z + 1} = \int \frac{2 dz}{(z - 1)^2} = -\frac{2}{z - 1} + C = -\frac{2}{\tan \frac{\theta}{2} - 1} + C$$

$$= \frac{2}{1 - \tan \frac{\theta}{2}} + C$$

$$37. \quad \int \frac{dt}{1 + \sin t + \cos t} = \int \frac{\frac{2 dz}{1 + z^2}}{1 + \frac{2z}{1 + z^2} + \frac{1 - z^2}{1 + z^2}} = \int \frac{dz}{z + 1} = \ln|z + 1| + C = \ln \left| \tan \frac{t}{2} + 1 \right| + C$$

$$38. \quad \int_0^{\pi/2} \frac{d\theta}{2 + \cos \theta} = \int_0^1 \frac{\left(\frac{2 dz}{1 + z^2} \right)}{2 + \left(\frac{1 - z^2}{1 + z^2} \right)} = \int_0^1 \frac{2 dz}{2 + 2z^2 + 1 - z^2} = \int_0^1 \frac{2 dz}{z^2 + 3} = \frac{2}{\sqrt{3}} \left[\tan^{-1} \frac{z}{\sqrt{3}} \right]_0^1 = \frac{2}{\sqrt{3}} \tan^{-1} \frac{1}{\sqrt{3}}$$

$$= \frac{\pi}{3\sqrt{3}} = \frac{\sqrt{3}\pi}{9}$$

$$39. \quad \int_{\pi/2}^{2\pi/3} \frac{\cos \theta d\theta}{\sin \theta \cos \theta + \sin \theta} = \int_1^{\sqrt{3}} \frac{\left(\frac{1 - z^2}{1 + z^2} \right) \left(\frac{2 dz}{1 + z^2} \right)}{\left[\frac{2z(1 - z^2)}{(1 + z^2)^2} + \left(\frac{2z}{1 + z^2} \right) \right]} = \int_1^{\sqrt{3}} \frac{2(1 - z^2) dz}{2z - 2z^3 + 2z + 2z^3} = \int_1^{\sqrt{3}} \frac{1 - z^2}{2z} dz$$

$$= \left[\frac{1}{2} \ln z - \frac{z^2}{4} \right]_1^{\sqrt{3}} = \left(\frac{1}{2} \ln \sqrt{3} - \frac{3}{4} \right) - \left(0 - \frac{1}{4} \right) = \frac{\ln 3}{4} - \frac{1}{2} = \frac{1}{4} (\ln 3 - 2) = \frac{1}{2} (\ln \sqrt{3} - 1)$$

$$\begin{aligned} 40. \int \frac{dt}{\sin t - \cos t} &= \int \frac{\left(\frac{2 dz}{1+z^2} \right)}{\left(\frac{2z}{1+z^2} - \frac{1-z^2}{1+z^2} \right)} = \int \frac{2 dz}{2z-1+z^2} = \int \frac{2 dz}{(z+1)^2-2} = \frac{1}{\sqrt{2}} \ln \left| \frac{z+1-\sqrt{2}}{z+1+\sqrt{2}} \right| + C \\ &= \frac{1}{\sqrt{2}} \ln \left| \frac{\tan\left(\frac{t}{2}\right) + 1 - \sqrt{2}}{\tan\left(\frac{t}{2}\right) + 1 + \sqrt{2}} \right| + C \end{aligned}$$

$$\begin{aligned} 41. \int \frac{\cos t \, dt}{1 - \cos t} &= \int \frac{\left(\frac{1-z^2}{1+z^2} \right) \left(\frac{2 dz}{1+z^2} \right)}{1 - \left(\frac{1-z^2}{1+z^2} \right)} = \int \frac{2(1-z^2) dz}{(1+z^2)^2 - (1+z^2)(1-z^2)} = \int \frac{2(1-z^2) dz}{(1+z^2)(1+z^2-1+z^2)} \\ &= \int \frac{(1-z^2) dz}{(1+z^2)z^2} = \int \frac{dz}{z^2(1+z^2)} - \int \frac{dz}{1+z^2} = \int \frac{dz}{z^2} - 2 \int \frac{dz}{z^2+1} = -\frac{1}{z} - 2 \tan^{-1} z + C = -\cot\left(\frac{t}{2}\right) - t + C \end{aligned}$$

7.5 INTEGRAL TABLES, COMPUTER ALGEBRA SYSTEMS, AND MONTE CARLO INTEGRATION

$$1. \int \frac{dx}{x\sqrt{x-3}} = \frac{2}{\sqrt{3}} \tan^{-1} \sqrt{\frac{x-3}{3}} + C$$

(We used FORMULA 13(a) with $a = 1$, $b = -3$)

$$2. \int \frac{x \, dx}{\sqrt{x-2}} = \int \frac{(x-2) \, dx}{\sqrt{x-2}} + 2 \int \frac{dx}{\sqrt{x-2}} = \int (\sqrt{x-2})^1 \, dx + 2 \int (\sqrt{x-2})^{-1} \, dx$$

$$= \left(\frac{2}{1} \right) \frac{(\sqrt{x-2})^3}{3} + 2 \left(\frac{2}{1} \right) \frac{(\sqrt{x-2})^1}{1} = \sqrt{x-2} \left[\frac{2(x-2)}{3} + 4 \right] + C$$

(We used FORMULA 11 with $a = 1$, $b = -2$, $n = 1$ and $a = 1$, $b = -2$, $n = -1$)

$$3. \int x\sqrt{2x-3} \, dx = \frac{1}{2} \int (2x-3)\sqrt{2x-3} \, dx + \frac{3}{2} \int \sqrt{2x-3} \, dx = \frac{1}{2} \int (\sqrt{2x-3})^3 \, dx + \frac{3}{2} \int (\sqrt{2x-3})^1 \, dx$$

$$= \left(\frac{1}{2} \right) \left(\frac{2}{2} \right) \frac{(\sqrt{2x-3})^5}{5} + \left(\frac{3}{2} \right) \left(\frac{2}{2} \right) \frac{(\sqrt{2x-3})^3}{3} + C = \frac{(2x-3)^{3/2}}{2} \left[\frac{2x-3}{5} + 1 \right] + C = \frac{(2x-3)^{3/2}(x+1)}{5} + C$$

(We used FORMULA 11 with $a = 2$, $b = -3$, $n = 3$ and $a = 2$, $b = -3$, $n = 1$)

$$4. \int \frac{\sqrt{9-4x}}{x^2} \, dx = -\frac{\sqrt{9-4x}}{x} + \frac{(-4)}{2} \int \frac{dx}{x\sqrt{9-4x}} + C$$

(We used FORMULA 14 with $a = -4$, $b = 9$)

$$= -\frac{\sqrt{9-4x}}{x} - 2\left(\frac{1}{\sqrt{9}}\right) \ln \left| \frac{\sqrt{9-4x}-\sqrt{9}}{\sqrt{9-4x}+\sqrt{9}} \right| + C$$

(We used FORMULA 13(b) with $a = -4$, $b = 9$)

$$= -\frac{\sqrt{9-4x}}{x} - \frac{2}{3} \ln \left| \frac{\sqrt{9-4x}-3}{\sqrt{9-4x}+3} \right| + C$$

$$\begin{aligned} 5. \int x\sqrt{4x-x^2} \, dx &= \int x\sqrt{2 \cdot 2x-x^2} \, dx = \frac{(x+2)(2x-3 \cdot 2)\sqrt{2 \cdot 2x-x^2}}{6} + \frac{2^3}{2} \sin^{-1}\left(\frac{x-2}{2}\right) + C \\ &= \frac{(x+2)(2x-6)\sqrt{4x-x^2}}{6} + 4 \sin^{-1}\left(\frac{x-2}{2}\right) + C \end{aligned}$$

(We used FORMULA 51 with $a = 2$)

$$6. \int \frac{dx}{x\sqrt{7+x^2}} = \int \frac{dx}{x\sqrt{(\sqrt{7})^2+x^2}} = -\frac{1}{\sqrt{7}} \ln \left| \frac{\sqrt{7} + \sqrt{(\sqrt{7})^2+x^2}}{x} \right| + C = -\frac{1}{\sqrt{7}} \ln \left| \frac{\sqrt{7} + \sqrt{7+x^2}}{x} \right| + C$$

(We used FORMULA 26 with $a = \sqrt{7}$)

$$7. \int \frac{\sqrt{4-x^2}}{x} \, dx = \int \frac{\sqrt{2^2-x^2}}{x} \, dx = \sqrt{2^2-x^2} - 2 \ln \left| \frac{2 + \sqrt{2^2-x^2}}{x} \right| + C = \sqrt{4-x^2} - 2 \ln \left| \frac{2 + \sqrt{4-x^2}}{x} \right| + C$$

(We used FORMULA 31 with $a = 2$)

$$8. \int \sqrt{25-p^2} \, dp = \int \sqrt{5^2-p^2} \, dp = \frac{p}{2} \sqrt{5^2-p^2} + \frac{5^2}{2} \sin^{-1} \frac{p}{5} + C = \frac{p}{2} \sqrt{25-p^2} + \frac{25}{2} \sin^{-1} \frac{p}{5} + C$$

(We used FORMULA 29 with $a = 5$)

$$9. \int \frac{r^2}{\sqrt{4-r^2}} \, dr = \int \frac{r^2}{\sqrt{2^2-r^2}} \, dr = \frac{2^2}{2} \sin^{-1}\left(\frac{r}{2}\right) - \frac{1}{2} r \sqrt{2^2-r^2} + C = 2 \sin^{-1}\left(\frac{r}{2}\right) - \frac{1}{2} r \sqrt{4-r^2} + C$$

(We used FORMULA 33 with $a = 2$)

$$10. \int \frac{d\theta}{5+4 \sin 2\theta} = \frac{-2}{2\sqrt{25-16}} \tan^{-1} \left[\sqrt{\frac{5-4}{5+4}} \tan\left(\frac{\pi}{4} - \frac{2\theta}{2}\right) \right] + C = -\frac{1}{3} \tan^{-1} \left[\frac{1}{3} \tan\left(\frac{\pi}{4} - \theta\right) \right] + C$$

(We used FORMULA 70 with $b = 5$, $c = 4$, $a = 2$)

$$11. \int e^{2t} \cos 3t \, dt = \frac{e^{2t}}{2^2+3^2} (2 \cos 3t + 3 \sin 3t) + C = \frac{e^{2t}}{13} (2 \cos 3t + 3 \sin 3t) + C$$

(We used FORMULA 108 with $a = 2$, $b = 3$)

$$12. \int x \cos^{-1} x \, dx = \int x^1 \cos^{-1} x \, dx = \frac{x^{1+1}}{1+1} \cos^{-1} x + \frac{1}{1+1} \int \frac{x^{1+1} dx}{\sqrt{1-x^2}} = \frac{x^2}{2} \cos^{-1} x + \frac{1}{2} \int \frac{x^2 dx}{\sqrt{1-x^2}}$$

(We used FORMULA 100 with $a = 1$, $n = 1$)

$$= \frac{x^2}{2} \cos^{-1} x + \frac{1}{2} \left(\frac{1}{2} \sin^{-1} x \right) - \frac{1}{2} \left(\frac{1}{2} x \sqrt{1-x^2} \right) + C = \frac{x^2}{2} \cos^{-1} x + \frac{1}{4} \sin^{-1} x - \frac{1}{4} x \sqrt{1-x^2} + C$$

(We used FORMULA 33 with $a = 1$)

$$13. \int \frac{ds}{(9-s^2)^2} = \int \frac{ds}{(3^2-s^2)^2} = \frac{s}{2 \cdot 3^2 \cdot (3^2-s^2)} + \frac{1}{2 \cdot 3^2} \int \frac{ds}{3^2-s^2}$$

(We used FORMULA 19 with $a = 3$)

$$= \frac{s}{18(9-s^2)} + \frac{1}{18} \left(\frac{1}{2 \cdot 3} \ln \left| \frac{s+3}{s-3} \right| \right) + C = \frac{s}{18(9-s^2)} + \frac{1}{108} \ln \left| \frac{s+3}{s-3} \right| + C$$

(We used FORMULA 18 with $a = 3$)

$$14. \int \frac{\sqrt{4x+9}}{x^2} dx = -\frac{\sqrt{4x+9}}{x} + \frac{4}{2} \int \frac{dx}{x\sqrt{4x+9}}$$

(We used FORMULA 14 with $a = 4$, $b = 9$)

$$= -\frac{\sqrt{4x+9}}{x} + 2 \left(\frac{1}{\sqrt{9}} \ln \left| \frac{\sqrt{4x+9} - \sqrt{9}}{\sqrt{4x+9} + \sqrt{9}} \right| \right) + C = -\frac{\sqrt{4x+9}}{x} + \frac{2}{3} \ln \left| \frac{\sqrt{4x+9} - 3}{\sqrt{4x+9} + 3} \right| + C$$

(We used FORMULA 13(b) with $a = 4$, $b = 9$)

$$15. \int \frac{\sqrt{3t-4}}{t} dt = 2\sqrt{3t-4} + (-4) \int \frac{dt}{t\sqrt{3t-4}}$$

(We used FORMULA 12 with $a = 3$, $b = -4$)

$$= 2\sqrt{3t-4} - 4 \left(\frac{2}{\sqrt{4}} \tan^{-1} \sqrt{\frac{3t-4}{4}} \right) + C = 2\sqrt{3t-4} - 4 \tan^{-1} \sqrt{\frac{3t-4}{4}} + C$$

(We used FORMULA 13(a) with $a = 3$, $b = -4$)

$$16. \int x^2 \tan^{-1} x dx = \frac{x^{2+1}}{2+1} \tan^{-1} x - \frac{1}{2+1} \int \frac{x^{2+1}}{1+x^2} dx = \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} dx$$

(We used FORMULA 101 with $a = 1$, $n = 2$);

$$\int \frac{x^3}{1+x^2} dx = \int x dx - \int \frac{x dx}{1+x^2} = \frac{x^2}{2} - \frac{1}{2} \ln(1+x^2) + C \Rightarrow \int x^2 \tan^{-1} x dx = \frac{x^3}{3} \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \ln(1+x^2) + C$$

$$17. \int \sin 3x \cos 2x dx = -\frac{\cos 5x}{10} - \frac{\cos x}{2} + C$$

(We used FORMULA 62(a) with $a = 3$, $b = 2$)

$$18. \int 8 \sin 4t \sin \frac{t}{2} dx = \frac{8}{7} \sin \left(\frac{7t}{2} \right) - \frac{8}{9} \sin \left(\frac{9t}{2} \right) + C = 8 \left[\frac{\sin \left(\frac{7t}{2} \right)}{7} - \frac{\sin \left(\frac{9t}{2} \right)}{9} \right] + C$$

(We used FORMULA 62(b) with $a = 4$, $b = \frac{1}{2}$)

$$19. \int \cos \frac{\theta}{3} \cos \frac{\theta}{4} d\theta = 6 \sin\left(\frac{\theta}{12}\right) + \frac{6}{7} \sin\left(\frac{7\theta}{12}\right) + C$$

(We used FORMULA 62(c) with $a = \frac{1}{3}$, $b = \frac{1}{4}$)

$$20. \int \cos \frac{\theta}{2} \cos 7\theta d\theta = \frac{1}{13} \sin\left(\frac{13\theta}{2}\right) + \frac{1}{15} \sin\left(\frac{15\theta}{2}\right) + C = \frac{\sin\left(\frac{13\theta}{2}\right)}{13} + \frac{\sin\left(\frac{15\theta}{2}\right)}{15} + C$$

(We used FORMULA 62(c) with $a = \frac{1}{2}$, $b = 7$)

$$21. \int \frac{x^3 + x + 1}{(x^2 + 1)^2} dx = \int \frac{x dx}{x^2 + 1} + \int \frac{dx}{(x^2 + 1)^2} = \frac{1}{2} \int \frac{d(x^2 + 1)}{x^2 + 1} + \int \frac{dx}{(x^2 + 1)^2}$$

$$= \frac{1}{2} \ln|x^2 + 1| + \frac{x}{2(1 + x^2)} + \frac{1}{2} \tan^{-1} x + C$$

(For the second integral we used FORMULA 17 with $a = 1$)

$$22. \int \frac{x^2 + 6x}{(x^2 + 3)^2} dx = \int \frac{dx}{x^2 + 3} + \int \frac{6x dx}{(x^2 + 3)^2} - \int \frac{3 dx}{(x^2 + 3)^2} = \int \frac{dx}{x^2 + (\sqrt{3})^2} + 3 \int \frac{d(x^2 + 3)}{(x^2 + 3)^2} - 3 \int \frac{dx}{[x^2 + (\sqrt{3})^2]^2}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{3}{(x^2 + 3)} - 3\left(\frac{x}{2(\sqrt{3})^2((\sqrt{3})^2 + x^2)} + \frac{1}{2(\sqrt{3})^3} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right) + C$$

(For the first integral we used FORMULA 16 with $a = \sqrt{3}$; for the third integral we used FORMULA 17 with $a = \sqrt{3}$)

$$= \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{3}{x^2 + 3} - \frac{x}{2(x^2 + 3)} + C$$

$$23. \int \sin^{-1} \sqrt{x} dx; \left[\begin{array}{l} u = \sqrt{x} \\ x = u^2 \\ dx = 2u du \end{array} \right] \rightarrow 2 \int u^1 \sin^{-1} u du = 2 \left(\frac{u^{1+1}}{1+1} \sin^{-1} u - \frac{1}{1+1} \int \frac{u^{1+1}}{\sqrt{1-u^2}} du \right)$$

$$= u^2 \sin^{-1} u - \int \frac{u^2 du}{\sqrt{1-u^2}}$$

(We used FORMULA 99 with $a = 1$, $n = 1$)

$$= u^2 \sin^{-1} u - \left(\frac{1}{2} \sin^{-1} u - \frac{1}{2} u \sqrt{1-u^2} \right) + C = \left(u^2 - \frac{1}{2} \right) \sin^{-1} u + \frac{1}{2} u \sqrt{1-u^2} + C$$

(We used FORMULA 33 with $a = 1$)

$$= \left(x - \frac{1}{2} \right) \sin^{-1} \sqrt{x} + \frac{1}{2} \sqrt{x-x^2} + C$$

$$24. \int \frac{\cos^{-1} \sqrt{x}}{\sqrt{x}} dx; \left[\begin{array}{l} u = \sqrt{x} \\ x = u^2 \\ dx = 2u du \end{array} \right] \rightarrow \int \frac{\cos^{-1} u}{u} \cdot 2u du = 2 \int \cos^{-1} u du = 2 \left(u \cos^{-1} u - \frac{1}{1} \sqrt{1-u^2} \right) + C$$

(We used FORMULA 97 with $a = 1$)

$$= 2\left(\sqrt{x} \cos^{-1} \sqrt{x} - \sqrt{1-x}\right) + C$$

$$\begin{aligned} 25. \int (\cot t) \sqrt{1-\sin^2 t} \, dt &= \int \frac{\sqrt{1-\sin^2 t} (\cos t) \, dt}{\sin t}; \left[\begin{array}{l} u = \sin t \\ du = \cos t \, dt \end{array} \right] \rightarrow \int \frac{\sqrt{1-u^2} \, du}{u} \\ &= \sqrt{1-u^2} - \ln \left| \frac{1+\sqrt{1-u^2}}{u} \right| + C \end{aligned}$$

(We used FORMULA 31 with $a = 1$)

$$= \sqrt{1-\sin^2 t} - \ln \left| \frac{1+\sqrt{1-\sin^2 t}}{\sin t} \right| + C$$

$$26. \int \frac{dt}{(\tan t) \sqrt{4-\sin^2 t}} = \int \frac{\cos t \, dt}{(\sin t) \sqrt{4-\sin^2 t}}; \left[\begin{array}{l} u = \sin t \\ du = \cos t \, dt \end{array} \right] \rightarrow \int \frac{du}{u \sqrt{4-u^2}} = -\frac{1}{2} \ln \left| \frac{2+\sqrt{4-u^2}}{u} \right| + C$$

(We used FORMULA 34 with $a = 2$)

$$= -\frac{1}{2} \ln \left| \frac{2+\sqrt{4-\sin^2 t}}{\sin t} \right| + C$$

$$27. \int \frac{dy}{y \sqrt{3+(\ln y)^2}}; \left[\begin{array}{l} u = \ln y \\ y = e^u \\ dy = e^u \, du \end{array} \right] \rightarrow \int \frac{e^u \, du}{e^u \sqrt{3+u^2}} = \int \frac{du}{\sqrt{3+u^2}} = \ln |u + \sqrt{3+u^2}| + C$$

$$= \ln |\ln y + \sqrt{3+(\ln y)^2}| + C$$

(We used FORMULA 20 with $a = \sqrt{3}$)

$$28. \int \frac{\cos \theta \, d\theta}{\sqrt{5+\sin^2 \theta}}; \left[\begin{array}{l} u = \sin \theta \\ du = \cos \theta \, d\theta \end{array} \right] \rightarrow \int \frac{du}{\sqrt{5+u^2}} = \ln |u + \sqrt{5+u^2}| + C = \ln |\sin \theta + \sqrt{5+\sin^2 \theta}| + C$$

(We used FORMULA 20 with $a = \sqrt{5}$)

$$29. \int \frac{3 \, dr}{\sqrt{9r^2-1}}; \left[\begin{array}{l} u = 3r \\ du = 3 \, dr \end{array} \right] \rightarrow \int \frac{du}{\sqrt{u^2-1}} = \ln |u + \sqrt{u^2-1}| + C = \ln |3r + \sqrt{9r^2-1}| + C$$

(We used FORMULA 36 with $a = 1$)

$$30. \int \frac{3 \, dy}{\sqrt{1+9y^2}}; \left[\begin{array}{l} u = 3y \\ du = 3 \, dy \end{array} \right] \rightarrow \int \frac{du}{\sqrt{1+u^2}} = \ln |u + \sqrt{1+u^2}| + C = \ln |3y + \sqrt{1+9y^2}| + C$$

(We used FORMULA 20 with $a = 1$)

$$31. \int \cos^{-1} \sqrt{x} \, dx; \left[\begin{array}{l} t = \sqrt{x} \\ x = t^2 \\ dx = 2t \, dt \end{array} \right] \rightarrow 2 \int t \cos^{-1} t \, dt = 2 \left(\frac{t^2}{2} \cos^{-1} t + \frac{1}{2} \int \frac{t^2}{\sqrt{1-t^2}} \, dt \right) = t^2 \cos^{-1} t + \int \frac{t^2}{\sqrt{1-t^2}} \, dt$$

(We used FORMULA 100 with $a = 1$, $n = 1$)

$$= t^2 \cos^{-1} t + \frac{1}{2} \sin^{-1} t - \frac{1}{2} t \sqrt{1-t^2} + C$$

(We used FORMULA 33 with $a = 1$)

$$= x \cos^{-1} \sqrt{x} + \frac{1}{2} \sin^{-1} \sqrt{x} - \frac{1}{2} \sqrt{x} \sqrt{1-x} + C = x \cos^{-1} \sqrt{x} + \frac{1}{2} \sin^{-1} \sqrt{x} - \frac{1}{2} \sqrt{x-x^2} + C$$

$$32. \int \tan^{-1} \sqrt{y} \, dy; \left[\begin{array}{l} t = \sqrt{y} \\ y = t^2 \\ dy = 2t \, dt \end{array} \right] \rightarrow 2 \int t \tan^{-1} t \, dt = 2 \left[\frac{t^2}{2} \tan^{-1} t - \frac{1}{2} \int \frac{t^2}{1+t^2} \, dt \right] = t^2 \tan^{-1} t - \int \frac{t^2}{1+t^2} \, dt$$

(We used FORMULA 101 with $n = 1$, $a = 1$)

$$= t^2 \tan^{-1} t - \int \frac{t^2+1}{t^2+1} \, dt + \int \frac{dt}{1+t^2} = t^2 \tan^{-1} t - t + \tan^{-1} t + C = y \tan^{-1} \sqrt{y} + \tan^{-1} \sqrt{y} - \sqrt{y} + C$$

$$33. \int x e^{3x} \, dx = \frac{e^{3x}}{3^2} (3x-1) + C = \frac{e^{3x}}{9} (3x-1) + C$$

(We used FORMULA 104 with $a = 3$)

$$34. \int x^3 e^{x/2} \, dx = 2x^3 e^{x/2} - 3 \cdot 2 \int x^2 e^{x/2} \, dx = 2x^3 e^{x/2} - 6 \left(2x^2 e^{x/2} - 2 \cdot 2 \int x e^{x/2} \, dx \right) \\ = 2x^3 e^{x/2} - 12x^2 e^{x/2} + 24 \cdot 4 e^{x/2} \left(\frac{x}{2} - 1 \right) + C = 2x^3 e^{x/2} - 12x^2 e^{x/2} + 96 e^{x/2} \left(\frac{x}{2} - 1 \right) + C$$

(We used FORMULA 105 with $a = \frac{1}{2}$ twice and FORMULA 104 with $a = \frac{1}{2}$)

$$35. \int x^2 2^x \, dx = \frac{x^2 2^x}{\ln 2} - \frac{2}{\ln 2} \int x 2^x \, dx = \frac{x^2 2^x}{\ln 2} - \frac{2}{\ln 2} \left(\frac{x 2^x}{\ln 2} - \frac{1}{\ln 2} \int 2^x \, dx \right) = \frac{x^2 2^x}{\ln 2} - \frac{2}{\ln 2} \left[\frac{x 2^x}{\ln 2} - \frac{2^x}{(\ln 2)^2} \right] + C$$

(We used FORMULA 106 with $a = 1$, $b = 2$)

$$36. \int x \pi^x \, dx = \frac{x \pi^x}{\ln \pi} - \frac{1}{\ln \pi} \int \pi^x \, dx = \frac{x \pi^x}{\ln \pi} - \frac{1}{\ln \pi} \left(\frac{\pi^x}{\ln \pi} \right) + C = \frac{x \pi^x}{\ln \pi} - \frac{\pi^x}{\ln \pi} - \frac{\pi^x}{(\ln \pi)^2} + C$$

(We used FORMULA 106 with $n = 1$, $b = \pi$, $a = 1$)

$$37. \int \frac{1}{8} \sinh^5 3x \, dx = \frac{1}{8} \left(\frac{\sinh^4 3x \cosh 3x}{5 \cdot 3} - \frac{5-1}{5} \int \sinh^3 3x \, dx \right) \\ = \frac{\sinh^4 3x \cosh 3x}{120} - \frac{1}{10} \left(\frac{\sinh^2 3x \cosh 3x}{3 \cdot 3} - \frac{3-1}{3} \int \sinh 3x \, dx \right)$$

(We used FORMULA 117 with $a = 3$, $n = 5$ and $a = 1$, $n = 3$)

$$\begin{aligned}
&= \frac{\sinh^4 3x \cosh 3x}{120} - \frac{\sinh^2 3x \cosh 3x}{90} + \frac{2}{30} \left(\frac{1}{3} \cosh 3x \right) + C \\
&= \frac{1}{120} \sinh^4 3x \cosh 3x - \frac{1}{90} \sinh^2 3x \cosh 3x + \frac{2}{90} \cosh 3x + C
\end{aligned}$$

$$\begin{aligned}
38. \quad \int \frac{\cosh^4 \sqrt{x}}{\sqrt{x}} dx; \left[\begin{array}{l} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{array} \right] &\rightarrow 2 \int \cosh^4 u \, du = 2 \left(\frac{\cosh^3 u \sinh u}{4} + \frac{4-1}{4} \int \cosh^2 u \, du \right) \\
&= \frac{\cosh^3 u \sinh u}{2} + \frac{3}{2} \left(\frac{\sinh 2u}{4} + \frac{u}{2} \right) + C
\end{aligned}$$

(We used FORMULA 118 with $a = 1$, $n = 2$ and FORMULA 116 with $a = 1$)

$$= \frac{1}{2} \cosh^3 \sqrt{x} \sinh \sqrt{x} + \frac{3}{8} \sinh 2\sqrt{x} + \frac{3}{4} \sqrt{x} + C$$

$$39. \quad \int x^2 \cosh 3x \, dx = \frac{x^2}{3} \sinh 3x - \frac{2}{3} \int x \sinh 3x \, dx = \frac{x^2}{3} \sinh 3x - \frac{2}{3} \left(\frac{x}{3} \cosh 3x - \frac{1}{3} \int \cosh 3x \, dx \right)$$

(We used FORMULA 122 with $a = 3$, $n = 2$ and FORMULA 121 with $a = 3$, $n = 1$)

$$= \frac{x^2}{3} \sinh 3x - \frac{2x}{9} \cosh 3x + \frac{2}{27} \sinh 3x + C$$

$$40. \quad \int x \sinh 5x \, dx = \frac{x}{5} \cosh 5x - \frac{1}{25} \sinh 5x + C$$

(We used FORMULA 119 with $a = 5$, $n = 1$)

$$41. \quad u = ax + b \Rightarrow x = \frac{u-b}{a} \Rightarrow dx = \frac{du}{a};$$

$$\int \frac{x \, dx}{(ax+b)^2} = \int \frac{(u-b) \, du}{au^2} = \frac{1}{a^2} \int \left(\frac{1}{u} - \frac{b}{u^2} \right) du = \frac{1}{a^2} \left[\ln |u| + \frac{b}{u} \right] + C = \frac{1}{a^2} \left[\ln |ax+b| + \frac{b}{ax+b} \right] + C$$

$$42. \quad x = a \sin \theta \Rightarrow a^2 - x^2 = a^2 \cos^2 \theta \Rightarrow -2x \, dx = -2a^2 \cos \theta \sin \theta \, d\theta \Rightarrow dx = a \cos \theta \, d\theta;$$

$$\int \sqrt{a^2 - x^2} \, dx = \int a \cos \theta (a \cos \theta) \, d\theta = a^2 \int \cos^2 \theta \, d\theta = \frac{a^2}{2} \int (1 + \cos 2\theta) \, d\theta = \frac{a^2}{2} \left(\theta + \frac{\sin 2\theta}{2} \right) + C$$

$$= \frac{a^2}{2} (\theta + \cos \theta \sin \theta) + C = \frac{a^2}{2} (\theta + \sqrt{1 - \sin^2 \theta} \cdot \sin \theta) + C = \frac{a^2}{2} \left(\sin^{-1} \frac{x}{a} + \frac{\sqrt{a^2 - x^2}}{a} \cdot \frac{x}{a} \right) + C$$

$$= \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C$$

$$43. \quad \int x^n (\ln ax)^m \, dx = \int (\ln ax)^m d \left(\frac{x^{n+1}}{n+1} \right) = \frac{x^{n+1} (\ln ax)^m}{n+1} - \int \left(\frac{x^{n+1}}{n+1} \right) m (\ln ax)^{m-1} \left(\frac{1}{x} \right) dx$$

$$= \frac{x^{n+1} (\ln ax)^m}{n+1} - \frac{m}{n+1} \int x^n (\ln ax)^{m-1} \, dx$$

(We used integration by parts $\int u \, dv = uv - \int v \, du$ with $u = (\ln ax)^m$, $v = \frac{x^{n+1}}{n+1}$)

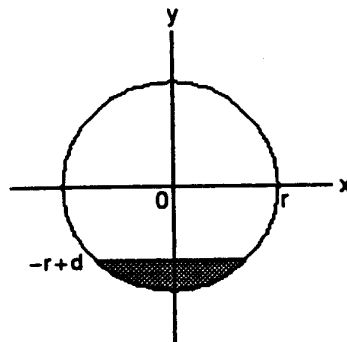
$$\begin{aligned}
 44. \int x^n \sin^{-1} ax \, dx &= \int \sin^{-1} ax \, d\left(\frac{x^{n+1}}{n+1}\right) = \frac{x^{n+1}}{n+1} \sin^{-1} ax - \int \left(\frac{x^{n+1}}{n+1}\right) \frac{a}{\sqrt{1-(ax)^2}} \, dx \\
 &= \frac{x^{n+1}}{n+1} \sin^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1} \, dx}{\sqrt{1-a^2 x^2}}, \quad n \neq -1
 \end{aligned}$$

(We used integration by parts $\int u \, dv = uv - \int v \, du$ with $u = \sin^{-1} ax$, $v = \frac{x^{n+1}}{n+1}$)

45. (a) The volume of the filled part equals the length of the tank times the area of the shaded region shown in the accompanying figure. Consider a layer of gasoline of thickness dy located at height y where $-r < y < -r+d$. The width of this layer is

$$2\sqrt{r^2 - y^2}. \text{ Therefore, } A = 2 \int_{-r}^{-r+d} \sqrt{r^2 - y^2} \, dy$$

$$\text{and } V = L \cdot A = 2L \int_{-r}^{-r+d} \sqrt{r^2 - y^2} \, dy$$



$$(b) \quad 2L \int_{-r}^{-r+d} \sqrt{r^2 - y^2} \, dy = 2L \left[\frac{y\sqrt{r^2 - y^2}}{2} + \frac{r^2}{2} \sin^{-1} \frac{y}{r} \right]_{-r}^{-r+d}$$

(We used FORMULA 29 with $a = r$)

$$= 2L \left[\frac{(d-r)}{2} \sqrt{2rd - d^2} + \frac{r^2}{2} \sin^{-1} \left(\frac{d-r}{r} \right) + \frac{r^2}{2} \left(\frac{\pi}{2} \right) \right] = 2L \left[\left(\frac{d-r}{2} \right) \sqrt{2rd - d^2} + \left(\frac{r^2}{2} \right) \left(\sin^{-1} \left(\frac{d-r}{r} \right) + \frac{\pi}{2} \right) \right]$$

46. The integrand $f(x) = \sqrt{x-x^2}$ is nonnegative, so the integral is maximized by integrating over the function's entire domain, which runs from $x = 0$ to $x = 1$

$$\Rightarrow \int_0^1 \sqrt{x-x^2} \, dx = \int_0^1 \sqrt{2 \cdot \frac{1}{2} x - x^2} \, dx = \left[\frac{\left(x - \frac{1}{2}\right)}{2} \sqrt{2 \cdot \frac{1}{2} x - x^2} + \frac{\left(\frac{1}{2}\right)^2}{2} \sin^{-1} \left(\frac{x - \frac{1}{2}}{\frac{1}{2}} \right) \right]_0^1$$

(We used FORMULA 48 with $a = \frac{1}{2}$)

$$= \left[\frac{\left(x - \frac{1}{2}\right)}{2} \sqrt{x-x^2} + \frac{1}{8} \sin^{-1} (2x-1) \right]_0^1 = \frac{1}{8} \cdot \frac{\pi}{2} - \frac{1}{8} \left(-\frac{\pi}{2} \right) = \frac{\pi}{8}$$

CAS EXPLORATIONS

For MAPLE use the `int(f(x),x)` command, and for MATHEMATICA use the command `Integrate[f(x),x]`, as discussed in the text.

$$47. (e) \int x^n \ln x \, dx = \frac{x^{n+1} \ln x}{n+1} - \frac{1}{n+1} \int x^n \, dx, \quad n \neq -1$$

(We used FORMULA 110 with $a = 1$, $m = 1$)

$$= \frac{x^{n+1} \ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C = \frac{x^{n+1}}{n+1} \left(\ln x - \frac{1}{n+1} \right) + C$$

$$48. (e) \int x^{-n} \ln x \, dx = \frac{x^{-n+1} \ln x}{-n+1} - \frac{1}{(-n)+1} \int x^{-n} \, dx, n \neq 1$$

(We used FORMULA 110 with $a = 1$, $m = 1$, $n = -n$)

$$= \frac{x^{1-n} \ln x}{1-n} - \frac{1}{1-n} \left(\frac{x^{1-n}}{1-n} \right) + C = \frac{x^{1-n}}{1-n} \left(\ln x - \frac{1}{1-n} \right) + C$$

49. (a) Neither MAPLE nor MATHEMATICA can find this integral for arbitrary n .

(b) MAPLE and MATHEMATICA get stuck at about $n = 5$.

(c) Let $x = \frac{\pi}{2} - u \Rightarrow dx = -du$; $x = 0 \Rightarrow u = \frac{\pi}{2}$, $x = \frac{\pi}{2} \Rightarrow u = 0$;

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{\sin^n x \, dx}{\sin^n x + \cos^n x} = \int_{\pi/2}^0 \frac{-\sin^n\left(\frac{\pi}{2} - u\right) du}{\sin^n\left(\frac{\pi}{2} - u\right) + \cos^n\left(\frac{\pi}{2} - u\right)} = \int_0^{\pi/2} \frac{\cos^n u \, du}{\cos^n u + \sin^n u} = \int_0^{\pi/2} \frac{\cos^n x \, dx}{\cos^n x + \sin^n x} \\ &\Rightarrow I + I = \int_0^{\pi/2} \left(\frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x} \right) dx = \int_0^{\pi/2} dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4} \end{aligned}$$

The following *Mathematica* module is used to obtain the Monte Carlo estimates of area in Problems 50 through 55.

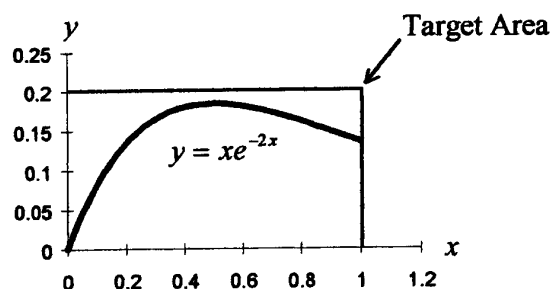
```
monte[f_, indvar_, m_, a_, b_, n_List] :=
Module[{g, x, xr, yr, area, lim, areaavg, y1, y2},
g = f/. indvar -> x;
lim = Length[n_List];
area = Table[0, {k, 1, lim}];
For[k = 1, k <= lim, k++,
For[counter = 0; i = 1, i <= n[[k]], i++,
xr = a + (b - a)*Random[];
yr = m*Random[];
If[yr <= g/. x -> xr, counter = counter + 1];];
area[[k]] = m*(b - a)*counter/n[[k]];
areaavg = (Sum[n[[i]]*area[[i]], {i, 1, lim}]) /
Sum[n[[i]], {i, 1, lim}];
y1 = Integrate[g, {x, a, b}] // N;
y2 = Integrate[g, {x, a, b}];
Print[area];
Print[areaavg];
Print[y2];
Print y1 ;
```

The following command executes the preceding module. The arguments are the integrand function, the independent variable, an upper bound on the integrand function, the lower limit of integration, the upper limit of integration, and a list of the numbers of random points to generate in each estimation.

```
monte[z*Sqrt[1 - z], z, 0.5, 0, 1, {100, 200, 300, 400,
500, 600, 700, 800, 900, 1000, 2000, 3000, 4000,
5000, 6000, 8000, 10000, 15000, 20000, 30000}
```

The preceding command is for Problem 51.

50.

Select $M = 0.2$

The area approximations will vary depending on the random number generator and seed value that is used

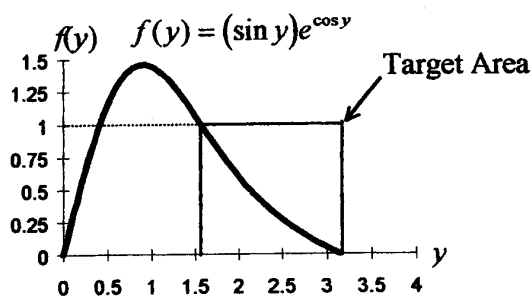
Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.154	2000	0.1492
200	0.151	3000	0.147867
300	0.148	4000	0.1497
400	0.149	5000	0.14712
500	0.1528	6000	0.148433
600	0.151667	8000	0.147925
700	0.149429	10,000	0.14796
800	0.1435	15,000	0.148147
900	0.146444	20,000	0.14824
1000	0.1408	30,000	0.147687

A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_0^1 xe^{-2x} dx \approx \left(\sum_{i=1}^{20} n_i \cdot \text{area}(i) \right) / \left(\sum_{i=1}^{20} n(i) \right) = 0.147987 \text{ by Monte Carlo.}$$

The actual value of the integral is $\frac{(1 - 3e^2)}{4} \approx 0.148499$.

51.

Select $M = 1$

The area approximations will vary depending on the random number generator and seed value that is used

Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.722566	2000	0.628319
200	0.628319	3000	0.646121
300	0.586431	4000	0.642456
400	0.581195	5000	0.636487
500	0.637743	6000	0.627533
600	0.560251	8000	0.643437
700	0.583439	10,000	0.62235
800	0.577268	15,000	0.625386
900	0.5621337	20,000	0.635073
1000	0.655022	30,000	0.638895