

CHAPTER 6 TRANSCENDENTAL FUNCTIONS AND DIFFERENTIAL EQUATIONS

6.1 LOGARITHMS

$$1. y = \ln 3x \Rightarrow y' = \left(\frac{1}{3x}\right)(3) = \frac{1}{x}$$

$$2. y = \ln(\theta + 1) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\theta + 1}\right)(1) = \frac{1}{\theta + 1}$$

$$3. y = \ln x^3 \Rightarrow \frac{dy}{dx} = \left(\frac{1}{x^3}\right)(3x^2) = \frac{3}{x}$$

$$4. y = (\ln x)^3 \Rightarrow \frac{dy}{dx} = 3(\ln x)^2 \cdot \frac{d}{dx}(\ln x) = \frac{3(\ln x)^2}{x}$$

$$5. y = t(\ln t)^2 \Rightarrow \frac{dy}{dt} = (\ln t)^2 + 2t(\ln t) \cdot \frac{d}{dt}(\ln t) = (\ln t)^2 + \frac{2t \ln t}{t} = (\ln t)^2 + 2 \ln t$$

$$6. y = t\sqrt{\ln t} = t(\ln t)^{1/2} \Rightarrow \frac{dy}{dt} = (\ln t)^{1/2} + \frac{1}{2}t(\ln t)^{-1/2} \cdot \frac{d}{dt}(\ln t) = (\ln t)^{1/2} + \frac{t(\ln t)^{-1/2}}{2t} \\ = (\ln t)^{1/2} + \frac{1}{2(\ln t)^{1/2}}$$

$$7. y = \frac{x^4}{4} \ln x - \frac{x^4}{16} \Rightarrow \frac{dy}{dx} = x^3 \ln x + \frac{x^4}{4} \cdot \frac{1}{x} - \frac{4x^3}{16} = x^3 \ln x$$

$$8. y = \frac{1 + \ln t}{t} \Rightarrow \frac{dy}{dt} = \frac{t\left(\frac{1}{t}\right) - (1 + \ln t)(1)}{t^2} = \frac{1 - 1 - \ln t}{t^2} = -\frac{\ln t}{t^2}$$

$$9. y = \frac{\ln t}{t} \Rightarrow \frac{dy}{dt} = \frac{t\left(\frac{1}{t}\right) - (\ln t)(1)}{t^2} = \frac{1 - \ln t}{t^2}$$

$$10. y = \frac{x \ln x}{1 + \ln x} \Rightarrow y' = \frac{(1 + \ln x)\left(\ln x + x \cdot \frac{1}{x}\right) - (x \ln x)\left(\frac{1}{x}\right)}{(1 + \ln x)^2} = \frac{(1 + \ln x)^2 - \ln x}{(1 + \ln x)^2} = 1 - \frac{\ln x}{(1 + \ln x)^2}$$

$$11. y = \frac{\ln x}{1 + \ln x} \Rightarrow y' = \frac{(1 + \ln x)\left(\frac{1}{x}\right) - (\ln x)\left(\frac{1}{x}\right)}{(1 + \ln x)^2} = \frac{\frac{1}{x} + \frac{\ln x}{x} - \frac{\ln x}{x}}{(1 + \ln x)^2} = \frac{1}{x(1 + \ln x)^2}$$

$$12. y = \ln(\ln(\ln x)) \Rightarrow y' = \frac{1}{\ln(\ln x)} \cdot \frac{d}{dx}(\ln(\ln x)) = \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{d}{dx}(\ln x) = \frac{1}{x(\ln x) \ln(\ln x)}$$

$$13. y = \theta[\sin(\ln \theta) + \cos(\ln \theta)] \Rightarrow \frac{dy}{d\theta} = [\sin(\ln \theta) + \cos(\ln \theta)] + \theta \left[\cos(\ln \theta) \cdot \frac{1}{\theta} - \sin(\ln \theta) \cdot \frac{1}{\theta} \right] \\ = \sin(\ln \theta) + \cos(\ln \theta) + \cos(\ln \theta) - \sin(\ln \theta) = 2 \cos(\ln \theta)$$

$$14. y = \ln(\sec \theta + \tan \theta) \Rightarrow \frac{dy}{d\theta} = \frac{\sec \theta \tan \theta + \sec^2 \theta}{\sec \theta + \tan \theta} = \frac{\sec \theta (\tan \theta + \sec \theta)}{\tan \theta + \sec \theta} = \sec \theta$$

$$15. y = \ln \frac{1}{x\sqrt{x+1}} = -\ln x - \frac{1}{2} \ln(x+1) \Rightarrow y' = -\frac{1}{x} - \frac{1}{2} \left(\frac{1}{x+1} \right) = -\frac{2(x+1) + x}{2x(x+1)} = -\frac{3x+2}{2x(x+1)}$$

$$16. y = \sqrt{\ln \sqrt{t}} = (\ln t^{1/2})^{1/2} \Rightarrow \frac{dy}{dt} = \frac{1}{2} (\ln t^{1/2})^{-1/2} \cdot \frac{d}{dt} (\ln t^{1/2}) = \frac{1}{2} (\ln t^{1/2})^{-1/2} \cdot \frac{1}{t^{1/2}} \cdot \frac{d}{dt} (t^{1/2})$$

$$= \frac{1}{2} (\ln t^{1/2})^{-1/2} \cdot \frac{1}{t^{1/2}} \cdot \frac{1}{2} t^{-1/2} = \frac{1}{4t\sqrt{\ln \sqrt{t}}}$$

$$17. y = \frac{1 + \ln t}{1 - \ln t} \Rightarrow \frac{dy}{dt} = \frac{(1 - \ln t) \left(\frac{1}{t} \right) - (1 + \ln t) \left(-\frac{1}{t} \right)}{(1 - \ln t)^2} = \frac{\frac{1}{t} - \frac{\ln t}{t} + \frac{1}{t} + \frac{\ln t}{t}}{(1 - \ln t)^2} = \frac{2}{t(1 - \ln t)^2}$$

$$18. y = \frac{1 + (\ln t)^2}{1 - (\ln t)^2} \Rightarrow \frac{dy}{dt} = \frac{(1 - (\ln t)^2) \left(\frac{2 \ln t}{t} \right) - (1 + (\ln t)^2) \left(-\frac{2 \ln t}{t} \right)}{(1 - (\ln t)^2)^2} = \frac{4 \ln t}{t(1 - (\ln t)^2)^2}$$

$$19. y = \ln(\sec(\ln \theta)) \Rightarrow \frac{dy}{d\theta} = \frac{1}{\sec(\ln \theta)} \cdot \frac{d}{d\theta} (\sec(\ln \theta)) = \frac{\sec(\ln \theta) \tan(\ln \theta)}{\sec(\ln \theta)} \cdot \frac{d}{d\theta} (\ln \theta) = \frac{\tan(\ln \theta)}{\theta}$$

$$20. y = \ln \left(\frac{(x^2 + 1)^5}{\sqrt{1-x}} \right) = 5 \ln(x^2 + 1) - \frac{1}{2} \ln(1-x) \Rightarrow y' = \frac{5 \cdot 2x}{x^2 + 1} - \frac{1}{2} \left(\frac{1}{1-x} \right) (-1) = \frac{10x}{x^2 + 1} + \frac{1}{2(1-x)}$$

$$21. y = \log_2 5\theta = \frac{\ln 5\theta}{\ln 2} \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\ln 2} \right) \left(\frac{1}{5\theta} \right) (5) = \frac{1}{\theta \ln 2}$$

$$22. y = \log_4 x + \log_4 x^2 = \frac{\ln x}{\ln 4} + \frac{\ln x^2}{\ln 4} = \frac{\ln x}{\ln 4} + 2 \frac{\ln x}{\ln 4} = 3 \frac{\ln x}{\ln 4} \Rightarrow y' = \frac{3}{x \ln 4}$$

$$23. y = \log_2 r \cdot \log_4 r = \left(\frac{\ln r}{\ln 2} \right) \left(\frac{\ln r}{\ln 4} \right) = \frac{\ln^2 r}{(\ln 2)(\ln 4)} \Rightarrow \frac{dy}{dr} = \left[\frac{1}{(\ln 2)(\ln 4)} \right] (2 \ln r) \left(\frac{1}{r} \right) = \frac{2 \ln r}{r(\ln 2)(\ln 4)}$$

$$24. y = \log_3 \left(\left(\frac{x+1}{x-1} \right)^{\ln 3} \right) = \frac{\ln \left(\frac{x+1}{x-1} \right)^{\ln 3}}{\ln 3} = \frac{(\ln 3) \ln \left(\frac{x+1}{x-1} \right)}{\ln 3} = \ln \left(\frac{x+1}{x-1} \right) = \ln(x+1) - \ln(x-1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x-1} = \frac{-2}{(x+1)(x-1)}$$

$$25. y = \theta \sin(\log_7 \theta) = \theta \sin \left(\frac{\ln \theta}{\ln 7} \right) \Rightarrow \frac{dy}{d\theta} = \sin \left(\frac{\ln \theta}{\ln 7} \right) + \theta \left[\cos \left(\frac{\ln \theta}{\ln 7} \right) \right] \left(\frac{1}{\theta \ln 7} \right) = \sin(\log_7 \theta) + \frac{1}{\ln 7} \cos(\log_7 \theta)$$

$$26. y = 3 \log_8 (\log_2 t) = \frac{3 \ln(\log_2 t)}{\ln 8} = \frac{3 \ln \left(\frac{\ln t}{\ln 2} \right)}{\ln 8} \Rightarrow \frac{dy}{dt} = \left(\frac{3}{\ln 8} \right) \left[\frac{1}{(\ln t)/(\ln 2)} \right] \left(\frac{1}{t \ln 2} \right) = \frac{3}{t(\ln t)(\ln 8)}$$

$$= \frac{1}{t(\ln t)(\ln 2)}$$

$$27. y = \int_{x^{2/2}}^{x^2} \ln \sqrt{t} \, dt \Rightarrow \frac{dy}{dx} = (\ln \sqrt{x^2}) \cdot \frac{d}{dx}(x^2) - \left(\ln \sqrt{\frac{x^2}{2}} \right) \cdot \frac{d}{dx}\left(\frac{x^2}{2}\right) = 2x \ln |x| - x \ln \frac{|x|}{\sqrt{2}}$$

$$28. y = \int_{\sqrt{x}}^{\sqrt[3]{x}} \ln t \, dt \Rightarrow \frac{dy}{dx} = (\ln \sqrt[3]{x}) \cdot \frac{d}{dx}(\sqrt[3]{x}) - (\ln \sqrt{x}) \cdot \frac{d}{dx}(\sqrt{x}) = (\ln \sqrt[3]{x})\left(\frac{1}{3}x^{-2/3}\right) - (\ln \sqrt{x})\left(\frac{1}{2}x^{-1/2}\right)$$

$$= \frac{\ln \sqrt[3]{x}}{3\sqrt[3]{x^2}} - \frac{\ln \sqrt{x}}{2\sqrt{x}}$$

$$29. \int_{-3}^{-2} \frac{1}{x} \, dx = [\ln |x|]_{-3}^{-2} = \ln 2 - \ln 3 = \ln \frac{2}{3}$$

$$30. \int_{-1}^0 \frac{3}{3x-2} \, dx = [\ln |3x-2|]_{-1}^0 = \ln 2 - \ln 5 = \ln \frac{2}{5}$$

$$31. \int \frac{2y}{y^2-25} \, dy = \ln |y^2-25| + C$$

$$32. \int \frac{8r}{4r^2-5} \, dr = \ln |4r^2-5| + C$$

$$33. \int_0^{\pi} \frac{\sin t}{2-\cos t} \, dt = [\ln |2-\cos t|]_0^{\pi} = \ln 3 - \ln 1 = \ln 3; \text{ or let } u = 2 - \cos t \Rightarrow du = \sin t \, dt \text{ with } t = 0$$

$$\Rightarrow u = 1 \text{ and } t = \pi \Rightarrow u = 3 \Rightarrow \int_0^{\pi} \frac{\sin t}{2-\cos t} \, dt = \int_1^3 \frac{1}{u} \, du = [\ln |u|]_1^3 = \ln 3 - \ln 1 = \ln 3$$

$$34. \int_0^{\pi/3} \frac{4 \sin \theta}{1-4 \cos \theta} \, d\theta = [\ln |1-4 \cos \theta|]_0^{\pi/3} = \ln |1-2| = -\ln 3 = \ln \frac{1}{3}; \text{ or let } u = 1 - 4 \cos \theta \Rightarrow du = 4 \sin \theta \, d\theta$$

$$\text{with } \theta = 0 \Rightarrow u = -3 \text{ and } \theta = \frac{\pi}{3} \Rightarrow u = -1 \Rightarrow \int_0^{\pi/3} \frac{4 \sin \theta}{1-4 \cos \theta} \, d\theta = \int_{-3}^{-1} \frac{1}{u} \, du = [\ln |u|]_{-3}^{-1} = -\ln 3 = \ln \frac{1}{3}$$

$$35. \text{ Let } u = \ln x \Rightarrow du = \frac{1}{x} \, dx; x = 1 \Rightarrow u = 0 \text{ and } x = 2 \Rightarrow u = \ln 2;$$

$$\int_1^2 \frac{2 \ln x}{x} \, dx = \int_0^{\ln 2} 2u \, du = [u^2]_0^{\ln 2} = (\ln 2)^2$$

$$36. \text{ Let } u = \ln x \Rightarrow du = \frac{1}{x} \, dx; x = 2 \Rightarrow u = \ln 2 \text{ and } x = 4 \Rightarrow u = \ln 4;$$

$$\int_2^4 \frac{dx}{x \ln x} = \int_{\ln 2}^{\ln 4} \frac{1}{u} \, du = [\ln |u|]_{\ln 2}^{\ln 4} = \ln(\ln 4) - \ln(\ln 2) = \ln\left(\frac{\ln 4}{\ln 2}\right) = \ln\left(\frac{\ln 2^2}{\ln 2}\right) = \ln\left(\frac{2 \ln 2}{\ln 2}\right) = \ln 2$$

37. Let $u = \ln x \Rightarrow du = \frac{1}{x} dx$; $x = 2 \Rightarrow u = \ln 2$ and $x = 4 \Rightarrow u = \ln 4$;

$$\int_2^4 \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\ln 4} u^{-2} du = \left[-\frac{1}{u} \right]_{\ln 2}^{\ln 4} = -\frac{1}{\ln 4} + \frac{1}{\ln 2} = -\frac{1}{\ln 2^2} + \frac{1}{\ln 2} = -\frac{1}{2 \ln 2} + \frac{1}{\ln 2} = \frac{1}{2 \ln 2} = \frac{1}{\ln 4}$$

38. Let $u = \ln x \Rightarrow du = \frac{1}{x} dx$; $x = 2 \Rightarrow u = \ln 2$ and $x = 16 \Rightarrow u = \ln 16$;

$$\int_2^{16} \frac{dx}{2x\sqrt{\ln x}} = \frac{1}{2} \int_{\ln 2}^{\ln 16} u^{-1/2} du = \left[u^{1/2} \right]_{\ln 2}^{\ln 16} = \sqrt{\ln 16} - \sqrt{\ln 2} = \sqrt{4 \ln 2} - \sqrt{\ln 2} = 2\sqrt{\ln 2} - \sqrt{\ln 2} = \sqrt{\ln 2}$$

39. Let $u = 6 + 3 \tan t \Rightarrow du = 3 \sec^2 t dt$;

$$\int \frac{3 \sec^2 t}{6 + 3 \tan t} dt = \int \frac{du}{u} = \ln |u| + C = \ln |6 + 3 \tan t| + C$$

40. Let $u = 2 + \sec y \Rightarrow du = \sec y \tan y dy$;

$$\int \frac{\sec y \tan y}{2 + \sec y} dy = \int \frac{du}{u} = \ln |u| + C = \ln |2 + \sec y| + C$$

41. Let $u = \cos \frac{x}{2} \Rightarrow du = -\frac{1}{2} \sin \frac{x}{2} dx \Rightarrow -2 du = \sin \frac{x}{2} dx$; $x = 0 \Rightarrow u = 1$ and $x = \frac{\pi}{2} \Rightarrow u = \frac{1}{\sqrt{2}}$;

$$\int_0^{\pi/2} \tan \frac{x}{2} dx = \int_0^{\pi/2} \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} dx = -2 \int_1^{1/\sqrt{2}} \frac{du}{u} = \left[-2 \ln |u| \right]_1^{1/\sqrt{2}} = -2 \ln \frac{1}{\sqrt{2}} = 2 \ln \sqrt{2} = \ln 2$$

42. Let $u = \sin t \Rightarrow du = \cos t dt$; $t = \frac{\pi}{4} \Rightarrow u = \frac{1}{\sqrt{2}}$ and $t = \frac{\pi}{2} \Rightarrow u = 1$;

$$\int_{\pi/4}^{\pi/2} \cot t dt = \int_{\pi/4}^{\pi/2} \frac{\cos t}{\sin t} dt = \int_{1/\sqrt{2}}^1 \frac{du}{u} = \left[\ln |u| \right]_{1/\sqrt{2}}^1 = -\ln \frac{1}{\sqrt{2}} = \ln \sqrt{2}$$

43. Let $u = \sin \frac{\theta}{3} \Rightarrow du = \frac{1}{3} \cos \frac{\theta}{3} d\theta \Rightarrow 6 du = 2 \cos \frac{\theta}{3} d\theta$; $\theta = \frac{\pi}{2} \Rightarrow u = \frac{1}{2}$ and $\theta = \pi \Rightarrow u = \frac{\sqrt{3}}{2}$;

$$\int_{\pi/2}^{\pi} 2 \cot \frac{\theta}{3} d\theta = \int_{\pi/2}^{\pi} \frac{2 \cos \frac{\theta}{3}}{\sin \frac{\theta}{3}} d\theta = 6 \int_{1/2}^{\sqrt{3}/2} \frac{du}{u} = 6 \left[\ln |u| \right]_{1/2}^{\sqrt{3}/2} = 6 \left(\ln \frac{\sqrt{3}}{2} - \ln \frac{1}{2} \right) = 6 \ln \sqrt{3} = \ln 27$$

44. Let $u = \cos 3x \Rightarrow du = -3 \sin 3x dx \Rightarrow -2 du = 6 \sin 3x dx$; $x = 0 \Rightarrow u = 1$ and $x = \frac{\pi}{12} \Rightarrow u = \frac{1}{\sqrt{2}}$;

$$\int_0^{\pi/12} 6 \tan 3x dx = \int_0^{\pi/12} \frac{6 \sin 3x}{\cos 3x} dx = -2 \int_1^{1/\sqrt{2}} \frac{du}{u} = -2 \left[\ln |u| \right]_1^{1/\sqrt{2}} = -2 \ln \frac{1}{\sqrt{2}} - \ln 1 = 2 \ln \sqrt{2} = \ln 2$$

45. $\int \frac{dx}{2\sqrt{x} + 2x} = \int \frac{dx}{2\sqrt{x}(1 + \sqrt{x})}$; let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} dx$; $\int \frac{dx}{2\sqrt{x}(1 + \sqrt{x})} = \int \frac{du}{u} = \ln |u| + C$

$$= \ln |1 + \sqrt{x}| + C = \ln(1 + \sqrt{x}) + C$$

46. Let $u = \sec x + \tan x \Rightarrow du = (\sec x \tan x + \sec^2 x) dx = (\sec x)(\tan x + \sec x) dx \Rightarrow \sec x dx = \frac{du}{u}$;

$$\int \frac{\sec x dx}{\sqrt{\ln(\sec x + \tan x)}} = \int \frac{du}{u\sqrt{\ln u}} = \int (\ln u)^{-1/2} \cdot \frac{1}{u} du = 2(\ln u)^{1/2} + C = 2\sqrt{\ln(\sec x + \tan x)} + C$$

47. (a) $f(x) = \ln(\cos x) \Rightarrow f'(x) = -\frac{\sin x}{\cos x} = -\tan x = 0 \Rightarrow x = 0$; $f'(x) > 0$ for $-\frac{\pi}{4} \leq x < 0$ and $f'(x) < 0$ for

$$0 < x \leq \frac{\pi}{3} \Rightarrow \text{there is a relative maximum at } x = 0 \text{ with } f(0) = \ln(\cos 0) = \ln 1 = 0; f\left(-\frac{\pi}{4}\right) = \ln\left(\cos\left(-\frac{\pi}{4}\right)\right) \\ = \ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2} \ln 2 \text{ and } f\left(\frac{\pi}{3}\right) = \ln\left(\cos\left(\frac{\pi}{3}\right)\right) = \ln \frac{1}{2} = -\ln 2. \text{ Therefore, the absolute minimum occurs at}$$

$$x = \frac{\pi}{3} \text{ with } f\left(\frac{\pi}{3}\right) = -\ln 2 \text{ and the absolute maximum occurs at } x = 0 \text{ with } f(0) = 0.$$

(b) $f(x) = \cos(\ln x) \Rightarrow f'(x) = \frac{-\sin(\ln x)}{x} = 0 \Rightarrow x = 1$; $f'(x) > 0$ for $\frac{1}{2} \leq x < 1$ and $f'(x) < 0$ for $1 < x \leq 2$

$$\Rightarrow \text{there is a relative maximum at } x = 1 \text{ with } f(1) = \cos(\ln 1) = \cos 0 = 1; f\left(\frac{1}{2}\right) = \cos\left(\ln\left(\frac{1}{2}\right)\right)$$

$$= \cos(-\ln 2) = \cos(\ln 2) \text{ and } f(2) = \cos(\ln 2). \text{ Therefore, the absolute minimum occurs at } x = \frac{1}{2} \text{ and}$$

$$x = 2 \text{ with } f\left(\frac{1}{2}\right) = f(2) = \cos(\ln 2), \text{ and the absolute maximum occurs at } x = 1 \text{ with } f(1) = 1.$$

48. (a) $f(x) = x - \ln x \Rightarrow f'(x) = 1 - \frac{1}{x}$; if $x > 1$, then $f'(x) > 0$ which means that $f(x)$ is increasing

(b) $f(1) = 1 - \ln 1 = 1 \Rightarrow f(x) = x - \ln x > 0$, if $x > 1$ by part (a) $\Rightarrow \ln x < x$ if $x > 1$

49. $\int_1^5 (\ln 2x - \ln x) dx = \int_1^5 (-\ln x + \ln 2 + \ln x) dx = (\ln 2) \int_1^5 dx = (\ln 2)(5 - 1) = \ln 2^4 = \ln 16$

50. $A = \int_{-\pi/4}^0 -\tan x dx + \int_0^{\pi/3} \tan x dx = \int_{-\pi/4}^0 \frac{-\sin x}{\cos x} dx - \int_0^{\pi/3} \frac{\sin x}{\cos x} dx = [\ln |\cos x|]_{-\pi/4}^0 - [\ln |\cos x|]_0^{\pi/3}$
 $= \left(\ln 1 - \ln \frac{1}{\sqrt{2}}\right) - \left(\ln \frac{1}{2} - \ln 1\right) = \ln \sqrt{2} + \ln 2 = \frac{3}{2} \ln 2$

51. $\frac{dy}{dx} = 1 + \frac{1}{x}$ at $(1, 3) \Rightarrow y = x + \ln |x| + C$; $y = 3$ at $x = 1 \Rightarrow C = 2 \Rightarrow y = x + \ln |x| + 2$

52. $\frac{d^2y}{dx^2} = \sec^2 x \Rightarrow \frac{dy}{dx} = \tan x + C$ and $1 = \tan 0 + C \Rightarrow \frac{dy}{dx} = \tan x + 1 \Rightarrow y = \int (\tan x + 1) dx$
 $= \ln |\sec x| + x + C_1$ and $0 = \ln |\sec 0| + 0 + C_1 \Rightarrow C_1 = 0 \Rightarrow y = \ln |\sec x| + x$

53. $\int \frac{\log_{10} x}{x} dx = \int \left(\frac{\ln x}{\ln 10}\right)\left(\frac{1}{x}\right) dx; [u = \ln x \Rightarrow du = \frac{1}{x} dx]$
 $\Rightarrow \int \left(\frac{\ln x}{\ln 10}\right)\left(\frac{1}{x}\right) dx = \frac{1}{\ln 10} \int u du = \left(\frac{1}{\ln 10}\right)\left(\frac{1}{2}u^2\right) + C = \frac{(\ln x)^2}{2 \ln 10} + C$

$$54. \int_1^4 \frac{\ln 2 \log_2 x}{x} dx = \int_1^4 \left(\frac{\ln 2}{x} \right) \left(\frac{\ln x}{\ln 2} \right) dx = \int_1^4 \frac{\ln x}{x} dx = \left[\frac{1}{2} (\ln x)^2 \right]_1^4 = \frac{1}{2} [(\ln 4)^2 - (\ln 1)^2] = \frac{1}{2} (\ln 4)^2$$

$$= \frac{1}{2} (2 \ln 2)^2 = 2 (\ln 2)^2$$

$$55. \int_0^2 \frac{\log_2 (x+2)}{x+2} dx = \frac{1}{\ln 2} \int_0^2 [\ln (x+2)] \left(\frac{1}{x+2} \right) dx = \left(\frac{1}{\ln 2} \right) \left[\frac{(\ln (x+2))^2}{2} \right]_0^2 = \left(\frac{1}{\ln 2} \right) \left[\frac{(\ln 4)^2}{2} - \frac{(\ln 2)^2}{2} \right]$$

$$= \left(\frac{1}{\ln 2} \right) \left[\frac{4(\ln 2)^2}{2} - \frac{(\ln 2)^2}{2} \right] = \frac{3}{2} \ln 2$$

$$56. \int_0^9 \frac{2 \log_{10} (x+1)}{x+1} dx = \frac{2}{\ln 10} \int_0^9 \ln (x+1) \left(\frac{1}{x+1} \right) dx = \left(\frac{2}{\ln 10} \right) \left[\frac{(\ln (x+1))^2}{2} \right]_0^9 = \left(\frac{2}{\ln 10} \right) \left[\frac{(\ln 10)^2}{2} - \frac{(\ln 1)^2}{2} \right]$$

$$= \ln 10$$

$$57. \int \frac{dx}{x \log_{10} x} = \int \left(\frac{\ln 10}{\ln x} \right) \left(\frac{1}{x} \right) dx = (\ln 10) \int \left(\frac{1}{\ln x} \right) \left(\frac{1}{x} \right) dx; \left[u = \ln x \Rightarrow du = \frac{1}{x} dx \right]$$

$$\Rightarrow (\ln 10) \int \left(\frac{1}{\ln x} \right) \left(\frac{1}{x} \right) dx = (\ln 10) \int \frac{1}{u} du = (\ln 10) \ln |u| + C = (\ln 10) \ln |\ln x| + C$$

$$58. \int \frac{dx}{x (\log_8 x)^2} = \int \frac{dx}{x \left(\frac{\ln x}{\ln 8} \right)^2} = (\ln 8)^2 \int \frac{(\ln x)^{-2}}{x} dx = (\ln 8)^2 \frac{(\ln x)^{-1}}{-1} + C = -\frac{(\ln 8)^2}{\ln x} + C$$

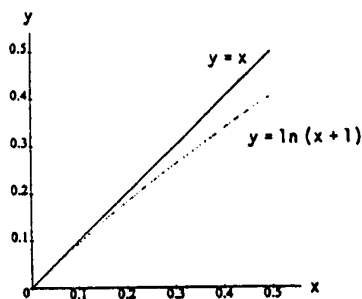
$$59. (a) L(x) = f(0) + f'(0) \cdot x, \text{ and } f(x) = \ln(1+x) \Rightarrow f'(x) \Big|_{x=0} = \frac{1}{1+x} \Big|_{x=0} = 1 \Rightarrow L(x) = \ln 1 + 1 \cdot x \Rightarrow L(x) = x$$

(b) On $[0, 0.1]$, $f(x)$ and $L(x)$ are both increasing because $f'(x) = \frac{1}{1+x} > 0$ and $L'(x) = 1 > 0$ for $0 \leq x \leq 0.1$. In

addition $0 \leq x \leq 0.1 \Rightarrow 1 \leq 1+x \leq 1.1 \Rightarrow \frac{1}{1.1} \leq \frac{1}{1+x} \leq 1 \Rightarrow L'(x) \geq f'(x) \Rightarrow E(x) = f(x) - L(x)$ is non-increasing on $[0, 0.1]$ because $E'(x) = f'(x) - L' \leq 0$ on the interval. Therefore, the largest error is

$$|f(0.1) - L(0.1)| = |\ln(1.1) - 0.1| \approx 0.00469.$$

- (c) The approximation $y = x$ for $\ln(1+x)$ is best for smaller positive values of x on the interval $[0, 0.1]$ as seen on the graph. As x increases so does the magnitude of the error $|\ln(x) - x|$. From the graph, an upper bound for the magnitude of the error is $|\ln(1.1) - 0.1| \approx 0.00469$ which is consistent with the analytical result obtained in part (b).



$$60. \ln(1.2) = \ln(1 + .2) \approx 0.2, \ln(.8) = \ln(1 + (-0.2)) \approx -0.2; \text{ with Simpson's rule for } n = 2, \ln(1.2) = \int_1^{1.2} \frac{1}{t} dt$$

$$\approx 0.182323232 \text{ and } \ln(0.8) = \int_1^{0.8} \frac{1}{t} dt \approx -0.223148148; \text{ alternatively, } \ln(1.2) = \ln(1 + 0.2) = \int_0^{0.2} \frac{1}{1+t} dt$$

$$\approx 0.182323232 \text{ and } \ln(0.8) = \int_0^{-0.2} \frac{1}{1+t} dt \approx -0.223148148.$$

6.2 EXPONENTIAL FUNCTIONS

1. $y = e^{-2x/3} \Rightarrow y' = -\frac{2}{3}e^{-2x/3}$
2. $y = e^{5-7x} \Rightarrow y' = e^{5-7x} \frac{d}{dx}(5-7x) \Rightarrow y' = -7e^{5-7x}$
3. $y = e^{(4\sqrt{x}+x^2)} \Rightarrow y' = e^{(4\sqrt{x}+x^2)} \frac{d}{dx}(4\sqrt{x}+x^2) \Rightarrow y' = \left(\frac{2}{\sqrt{x}} + 2x\right)e^{(4\sqrt{x}+x^2)}$
4. $y = (1+3x)e^{-x} \Rightarrow y' = (1+3x)(-e^{-x})3e^{-x} = (2-3x)e^{-x}$
5. $y = (x^2 - 2x + 2)e^x \Rightarrow y' = (2x-2)e^x + (x^2 - 2x + 2)e^x = x^2e^x$
6. $y = e^\theta(\sin \theta + \cos \theta) \Rightarrow y' = e^\theta(\sin \theta + \cos \theta) + e^\theta(\cos \theta - \sin \theta) = 2e^\theta \cos \theta$
7. $y = \ln(3\theta e^{-\theta}) = \ln 3 + \ln \theta + \ln e^{-\theta} = \ln 3 + \ln \theta - \theta \Rightarrow \frac{dy}{d\theta} = \frac{1}{\theta} - 1$
8. $y = \cos(e^{-\theta^2}) \Rightarrow \frac{dy}{d\theta} = -\sin(e^{-\theta^2}) \frac{d}{d\theta}(e^{-\theta^2}) = (-\sin(e^{-\theta^2}))(e^{-\theta^2}) \frac{d}{d\theta}(-\theta^2) = 2\theta e^{-\theta^2} \sin(e^{-\theta^2})$
9. $y = \ln(2e^{-t} \sin t) = \ln 2 + \ln e^{-t} + \ln \sin t = \ln 2 - t + \ln \sin t \Rightarrow \frac{dy}{dt} = -1 + \left(\frac{1}{\sin t}\right) \frac{d}{dt}(\sin t) = -1 + \frac{\cos t}{\sin t}$
 $= \frac{\cos t - \sin t}{\sin t}$
10. $y = \ln \frac{e^\theta}{1+e^\theta} = \ln e^\theta - \ln(1+e^\theta) = \theta - \ln(1+e^\theta) \Rightarrow \frac{dy}{d\theta} = 1 - \left(\frac{1}{1+e^\theta}\right) \frac{d}{d\theta}(1+e^\theta) = 1 - \frac{e^\theta}{1+e^\theta} = \frac{1}{1+e^\theta}$
11. $y = \ln \frac{\sqrt{\theta}}{1+\sqrt{\theta}} = \ln \sqrt{\theta} - \ln(1+\sqrt{\theta}) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\sqrt{\theta}}\right) \frac{d}{d\theta}(\sqrt{\theta}) - \left(\frac{1}{1+\sqrt{\theta}}\right) \frac{d}{d\theta}(1+\sqrt{\theta})$
 $= \left(\frac{1}{\sqrt{\theta}}\right)\left(\frac{1}{2\sqrt{\theta}}\right) - \left(\frac{1}{1+\sqrt{\theta}}\right)\left(\frac{1}{2\sqrt{\theta}}\right) = \frac{(1+\sqrt{\theta}) - \sqrt{\theta}}{2\theta(1+\sqrt{\theta})} = \frac{1}{2\theta(1+\sqrt{\theta})} = \frac{1}{2\theta(1+\theta^{1/2})}$
12. $y = e^{\sin t}(\ln t^2 + 1) \Rightarrow \frac{dy}{dt} = e^{\sin t}(\cos t)(\ln t^2 + 1) + \frac{2}{t}e^{\sin t} = e^{\sin t}\left[(\ln t^2 + 1)(\cos t) + \frac{2}{t}\right]$
13. $\int_0^{\ln x} \sin e^t dt \Rightarrow y' = (\sin e^{\ln x}) \cdot \frac{d}{dx}(\ln x) = \frac{\sin x}{x}$

$$\begin{aligned}
 14. \quad y &= \int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t \, dt \Rightarrow y' = (\ln e^{2x}) \cdot \frac{d}{dx}(e^{2x}) - (\ln e^{4\sqrt{x}}) \cdot \frac{d}{dx}(e^{4\sqrt{x}}) = (2x)(2e^{2x}) - (4\sqrt{x})(e^{4\sqrt{x}}) \cdot \frac{d}{dx}(4\sqrt{x}) \\
 &= 4xe^{2x} - 4\sqrt{x}e^{4\sqrt{x}} \left(\frac{2}{\sqrt{x}} \right) = 4xe^{2x} - 8e^{4\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 15. \quad \ln y &= e^y \sin x \Rightarrow \left(\frac{1}{y} \right) y' = (y' e^y)(\sin x) + e^y \cos x \Rightarrow y' \left(\frac{1}{y} - e^y \sin x \right) = e^y \cos x \\
 &\Rightarrow y' \left(\frac{1 - ye^y \sin x}{y} \right) = e^y \cos x \Rightarrow y' = \frac{ye^y \cos x}{1 - ye^y \sin x}
 \end{aligned}$$

$$\begin{aligned}
 16. \quad \ln xy &= e^{x+y} \Rightarrow \ln x + \ln y = e^{x+y} \Rightarrow \frac{1}{x} + \left(\frac{1}{y} \right) y' = (1 + y') e^{x+y} \Rightarrow y' \left(\frac{1}{y} - e^{x+y} \right) = e^{x+y} - \frac{1}{x} \\
 &\Rightarrow y' \left(\frac{1 - ye^{x+y}}{y} \right) = \frac{xe^{x+y} - 1}{x} \Rightarrow y' = \frac{y(xe^{x+y} - 1)}{x(1 - ye^{x+y})}
 \end{aligned}$$

$$\begin{aligned}
 17. \quad e^{2x} &= \sin(x + 3y) \Rightarrow 2e^{2x} = (1 + 3y') \cos(x + 3y) \Rightarrow 1 + 3y' = \frac{2e^{2x}}{\cos(x + 3y)} \Rightarrow 3y' = \frac{2e^{2x}}{\cos(x + 3y)} - 1 \\
 &\Rightarrow y' = \frac{2e^{2x} - \cos(x + 3y)}{3 \cos(x + 3y)}
 \end{aligned}$$

$$18. \quad \tan y = e^x + \ln x \Rightarrow (\sec^2 y) y' = e^x + \frac{1}{x} \Rightarrow y' = \frac{(xe^x + 1) \cos^2 y}{x}$$

$$19. \quad \int (e^{3x} + 5e^{-x}) \, dx = \frac{e^{3x}}{3} - 5e^{-x} + C$$

$$20. \quad \int_{\ln 2}^{\ln 3} e^x \, dx = [e^x]_{\ln 2}^{\ln 3} = e^{\ln 3} - e^{\ln 2} = 3 - 2 = 1$$

$$21. \quad \int 8e^{(x+1)} \, dx = 8e^{(x+1)} + C$$

$$22. \quad \int_{\ln 4}^{\ln 9} e^{x/2} \, dx = [2e^{x/2}]_{\ln 4}^{\ln 9} = 2[e^{(\ln 9)/2} - e^{(\ln 4)/2}] = 2(e^{\ln 3} - e^{\ln 2}) = 2(3 - 2) = 2$$

$$23. \quad \text{Let } u = -r^{1/2} \Rightarrow du = -\frac{1}{2}r^{-1/2} \, dr \Rightarrow -2 \, du = r^{-1/2} \, dr;$$

$$\int \frac{e^{-\sqrt{r}}}{\sqrt{r}} \, dr = \int e^{-r^{1/2}} \cdot r^{-1/2} \, dr = -2 \int e^u \, du = -2e^u + C = -2e^{-\sqrt{r}} + C$$

$$24. \quad \text{Let } u = -t^2 \Rightarrow du = -2t \, dt \Rightarrow -du = 2t \, dt;$$

$$\int 2te^{-t^2} \, dt = - \int e^u \, du = -e^u + C = -e^{-t^2} + C$$

25. Let $u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2} dx \Rightarrow -du = \frac{1}{x^2} dx$;

$$\int \frac{e^{1/x}}{x^2} dx = \int -e^u du = -e^u + C = -e^{1/x} + C$$

26. Let $u = -x^{-2} \Rightarrow du = 2x^{-3} dx \Rightarrow \frac{1}{2} du = x^{-3} dx$;

$$\int \frac{e^{-1/x^2}}{x^3} dx = \int e^{-x^{-2}} \cdot x^{-3} dx = \frac{1}{2} \int e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{-x^{-2}} + C = \frac{1}{2} e^{-1/x^2} + C$$

27. Let $u = \tan \theta \Rightarrow du = \sec^2 \theta d\theta$; $\theta = 0 \Rightarrow u = 0$, $\theta = \frac{\pi}{4} \Rightarrow u = 1$;

$$\begin{aligned} \int_0^{\pi/4} (1 + e^{\tan \theta}) \sec^2 \theta d\theta &= \int_0^{\pi/4} \sec^2 \theta d\theta + \int_0^1 e^u du = [\tan \theta]_0^{\pi/4} + [e^u]_0^1 = \left[\tan\left(\frac{\pi}{4}\right) - \tan(0) \right] + (e^1 - e^0) \\ &= (1 - 0) + (e - 1) = e \end{aligned}$$

28. Let $u = \cot \theta \Rightarrow du = -\csc^2 \theta d\theta$; $\theta = \frac{\pi}{4} \Rightarrow u = 1$, $\theta = \frac{\pi}{2} \Rightarrow u = 0$;

$$\begin{aligned} \int_{\pi/4}^{\pi/2} (1 + e^{\cot \theta}) \csc^2 \theta d\theta &= \int_{\pi/4}^{\pi/2} \csc^2 \theta d\theta - \int_1^0 e^u du = [-\cot \theta]_{\pi/4}^{\pi/2} - [e^u]_1^0 = \left[-\cot\left(\frac{\pi}{2}\right) + \cot\left(\frac{\pi}{4}\right) \right] - (e^0 - e^1) \\ &= (0 + 1) - (1 - e) = e \end{aligned}$$

29. Let $u = \sec \pi t \Rightarrow du = \frac{1}{\pi} \sec \pi t \tan \pi t dt \Rightarrow \pi du = \sec \pi t \tan \pi t dt$;

$$\int e^{\sec(\pi t)} \sec(\pi t) \tan(\pi t) dt = \frac{1}{\pi} \int e^u du = \frac{e^u}{\pi} + C = \frac{e^{\sec(\pi t)}}{\pi} + C$$

30. Let $u = e^{x^2} \Rightarrow du = 2xe^{x^2} dx$; $x = 0 \Rightarrow u = 1$, $x = \sqrt{\ln \pi} \Rightarrow u = e^{\ln \pi} = \pi$;

$$\int_0^{\sqrt{\ln \pi}} 2xe^{x^2} \cos(e^{x^2}) dx = \int_1^{\pi} \cos u du = [\sin u]_1^{\pi} = \sin(\pi) - \sin(1) = -\sin(1) \approx -0.84147$$

31. Let $u = 1 + e^r \Rightarrow du = e^r dr$;

$$\int \frac{e^r}{1 + e^r} dr = \int \frac{1}{u} du = \ln |u| + C = \ln(1 + e^r) + C$$

32. $\int \frac{1}{1 + e^x} dx = \int \frac{e^{-x}}{e^{-x} + 1} dx$;

let $u = e^{-x} + 1 \Rightarrow du = -e^{-x} dx \Rightarrow -du = e^{-x} dx$;

$$\int \frac{e^{-x}}{e^{-x} + 1} dx = - \int \frac{1}{u} du = -\ln |u| + C = -\ln(e^{-x} + 1) + C$$

$$33. y = 2^x \Rightarrow y' = 2^x \ln 2$$

$$34. y = 5\sqrt{s} \Rightarrow \frac{dy}{ds} = 5\sqrt{s} (\ln 5) \left(\frac{1}{2}s^{-1/2}\right) = \left(\frac{\ln 5}{2\sqrt{s}}\right) 5\sqrt{s}$$

$$35. y = x^\pi \Rightarrow y' = \pi x^{(\pi-1)}$$

$$36. y = (\cos \theta)^{\sqrt{2}} \Rightarrow \frac{dy}{d\theta} = -\sqrt{2} (\cos \theta)^{(\sqrt{2}-1)} (\sin \theta)$$

$$37. y = 7^{\sec \theta} \ln 7 \Rightarrow \frac{dy}{d\theta} = (7^{\sec \theta} \ln 7)(\ln 7)(\sec \theta \tan \theta) = 7^{\sec \theta} (\ln 7)^2 (\sec \theta \tan \theta)$$

$$38. y = 2^{\sin 3t} \Rightarrow \frac{dy}{dt} = (2^{\sin 3t} \ln 2)(\cos 3t)(3) = (3 \cos 3t)(2^{\sin 3t})(\ln 2)$$

$$39. y = t^{1-e} \Rightarrow \frac{dy}{dt} = (1-e)t^{-e}$$

$$40. y = (\ln \theta)^\pi \Rightarrow \frac{dy}{d\theta} = \pi (\ln \theta)^{(\pi-1)} \left(\frac{1}{\theta}\right) = \frac{\pi (\ln \theta)^{(\pi-1)}}{\theta}$$

$$41. y = \log_3 \left(\left(\frac{x+1}{x-1} \right)^{\ln 3} \right) = \frac{\ln \left(\frac{x+1}{x-1} \right)^{\ln 3}}{\ln 3} = \frac{(\ln 3) \ln \left(\frac{x+1}{x-1} \right)}{\ln 3} = \ln \left(\frac{x+1}{x-1} \right) = \ln(x+1) - \ln(x-1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x-1} = \frac{-2}{(x+1)(x-1)}$$

$$42. y = \log_5 \sqrt{\left(\frac{7x}{3x+2} \right)^{\ln 5}} = \log_5 \left(\frac{7x}{3x+2} \right)^{(\ln 5)/2} = \frac{\ln \left(\frac{7x}{3x+2} \right)^{(\ln 5)/2}}{\ln 5} = \left(\frac{\ln 5}{2} \right) \left[\frac{\ln \left(\frac{7x}{3x+2} \right)}{\ln 5} \right] = \frac{1}{2} \ln \left(\frac{7x}{3x+2} \right)$$

$$= \frac{1}{2} \ln 7x - \frac{1}{2} \ln(3x+2) \Rightarrow \frac{dy}{dx} = \frac{7}{2 \cdot 7x} - \frac{3}{2 \cdot (3x+2)} = \frac{(3x+2) - 3x}{2x(3x+2)} = \frac{1}{x(3x+2)}$$

$$43. y = \log_7 \left(\frac{\sin \theta \cos \theta}{e^{\theta 2^\theta}} \right) = \frac{\ln(\sin \theta) + \ln(\cos \theta) - \ln e^\theta - \ln 2^\theta}{\ln 7} = \frac{\ln(\sin \theta) + \ln(\cos \theta) - \theta - \theta \ln 2}{\ln 7}$$

$$\Rightarrow \frac{dy}{d\theta} = \frac{\cos \theta}{(\sin \theta)(\ln 7)} - \frac{\sin \theta}{(\cos \theta)(\ln 7)} - \frac{1}{\ln 7} - \frac{\ln 2}{\ln 7} = \left(\frac{1}{\ln 7} \right) (\cot \theta - \tan \theta - 1 - \ln 2)$$

$$44. y = \log_2 \left(\frac{x^2 e^2}{2\sqrt{x+1}} \right) = \frac{\ln x^2 + \ln e^2 - \ln 2 - \ln \sqrt{x+1}}{\ln 2} = \frac{2 \ln x + 2 - \ln 2 - \frac{1}{2} \ln(x+1)}{\ln 2}$$

$$\Rightarrow y' = \frac{2}{x \ln 2} - \frac{1}{2(\ln 2)(x+1)} = \frac{4(x+1) - x}{2x(x+1)(\ln 2)} = \frac{3x+4}{2x(x+1)\ln 2}$$

$$45. y = (x+1)^x \Rightarrow \ln y = \ln(x+1)^x = x \ln(x+1) \Rightarrow \frac{y'}{y} = \ln(x+1) + x \cdot \frac{1}{(x+1)} \Rightarrow y' = (x+1)^x \left[\frac{x}{x+1} + \ln(x+1) \right]$$

$$46. y = t^{\sqrt{t}} = t^{(t^{1/2})} \Rightarrow \ln y = \ln t^{(t^{1/2})} = (t^{1/2})(\ln t) \Rightarrow \frac{1}{y} \frac{dy}{dt} = \left(\frac{1}{2}t^{-1/2}\right)(\ln t) + t^{1/2}\left(\frac{1}{t}\right) = \frac{\ln t + 2}{2\sqrt{t}} \\ \Rightarrow \frac{dy}{dt} = \left(\frac{\ln t + 2}{2\sqrt{t}}\right)t^{\sqrt{t}}$$

$$47. y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x} = (\sin x)(\ln x) \Rightarrow \frac{y'}{y} = (\cos x)(\ln x) + (\sin x)\left(\frac{1}{x}\right) = \frac{\sin x + x(\ln x)(\cos x)}{x} \\ \Rightarrow y' = x^{\sin x} \left[\frac{\sin x + x(\ln x)(\cos x)}{x} \right]$$

$$48. y = (\ln x)^{\ln x} \Rightarrow \ln y = (\ln x) \ln(\ln x) \Rightarrow \frac{y'}{y} = \left(\frac{1}{x}\right) \ln(\ln x) + (\ln x) \left(\frac{1}{\ln x}\right) \frac{d}{dx}(\ln x) = \frac{\ln(\ln x)}{x} + \frac{1}{x} \\ \Rightarrow y' = \left(\frac{\ln(\ln x) + 1}{x}\right)(\ln x)^{\ln x}$$

$$49. \text{ Let } u = x^2 \Rightarrow du = 2x \, dx \Rightarrow \frac{1}{2} du = x \, dx; x = 1 \Rightarrow u = 1, x = \sqrt{2} \Rightarrow u = 2;$$

$$\int_1^{\sqrt{2}} x 2^{(x^2)} \, dx = \int_1^2 \left(\frac{1}{2}\right) 2^u \, du = \frac{1}{2} \left[\frac{2^u}{\ln 2} \right]_1^2 = \left(\frac{1}{2 \ln 2}\right)(2^2 - 2^1) = \frac{1}{\ln 2}$$

$$50. \text{ Let } u = \cos t \Rightarrow du = -\sin t \, dt \Rightarrow -du = \sin t \, dt; t = 0 \Rightarrow u = 1, t = \frac{\pi}{2} \Rightarrow u = 0;$$

$$\int_0^{\pi/2} 7^{\cos t} \sin t \, dt = - \int_1^0 7^u \, du = \left[-\frac{7^u}{\ln 7} \right]_1^0 = \left(\frac{-1}{\ln 7}\right)(7^0 - 7) = \frac{6}{\ln 7}$$

$$51. \text{ Let } u = \ln x \Rightarrow du = \frac{1}{x} \, dx; x = 1 \Rightarrow u = 0, x = 2 \Rightarrow u = \ln 2;$$

$$\int_1^2 \frac{2^{\ln x}}{x} \, dx = \int_0^{\ln 2} 2^u \, du = \left[\frac{2^u}{\ln 2} \right]_0^{\ln 2} = \left(\frac{1}{\ln 2}\right)(2^{\ln 2} - 2^0) = \frac{2^{\ln 2} - 1}{\ln 2}$$

$$52. \int 3x^{\sqrt{3}} \, dx = \frac{3x^{(\sqrt{3}+1)}}{\sqrt{3}+1} + C$$

$$53. \int x^{(\sqrt{2}-1)} \, dx = \frac{x^{\sqrt{2}}}{\sqrt{2}} + C$$

$$54. \int_0^3 (\sqrt{2}+1)x^{\sqrt{2}} \, dx = \left[\frac{x^{(\sqrt{2}+1)}}{(\sqrt{2}+1)} \right]_0^3 = \frac{3^{(\sqrt{2}+1)}}{(\sqrt{2}+1)}$$

$$55. \int_1^e x^{(\ln 2)-1} \, dx = \left[\frac{x^{\ln 2}}{\ln 2} \right]_1^e = \frac{e^{\ln 2} - 1^{\ln 2}}{\ln 2} = \frac{2-1}{\ln 2} = \frac{1}{\ln 2}$$

$$56. \text{ Let } u = \ln t \Rightarrow du = \frac{dt}{t}; u = 0 \text{ when } t = 1, \text{ and } u = x \text{ when } t = e^x.$$

$$\int_1^{e^x} \frac{3^{\ln t}}{t} \, dt = \int_0^x 3^u \, du = \left[\frac{3^u}{\ln 3} \right]_0^x = \frac{1}{\ln 3}(3^x - 1)$$

$$57. \frac{dy}{dt} = e^t \sin(e^t - 2) \Rightarrow y = \int e^t \sin(e^t - 2) \, dt;$$

$$\text{let } u = e^t - 2 \Rightarrow du = e^t \, dt \Rightarrow y = \int \sin u \, du = -\cos u + C = -\cos(e^t - 2) + C; y(\ln 2) = 0$$

$$\Rightarrow -\cos(e^{\ln 2} - 2) + C = 0 \Rightarrow -\cos(2 - 2) + C = 0 \Rightarrow C = \cos 0 = 1; \text{ thus, } y = 1 - \cos(e^t - 2)$$

$$58. \frac{dy}{dt} = e^{-t} \sec^2(\pi e^{-t}) \Rightarrow y = \int e^{-t} \sec^2(\pi e^{-t}) dt;$$

$$\begin{aligned} \text{let } u = \pi e^{-t} &\Rightarrow du = -\pi e^{-t} dt \Rightarrow -\frac{1}{\pi} du = e^{-t} dt \Rightarrow y = -\frac{1}{\pi} \int \sec^2 u du = -\frac{1}{\pi} \tan u + C \\ &= -\frac{1}{\pi} \tan(\pi e^{-t}) + C; y(\ln 4) = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi} \tan(\pi e^{-\ln 4}) + C = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi} \tan\left(\pi \cdot \frac{1}{4}\right) + C = \frac{2}{\pi} \\ &\Rightarrow -\frac{1}{\pi}(1) + C = \frac{2}{\pi} \Rightarrow C = \frac{3}{\pi}; \text{ thus, } y = \frac{3}{\pi} - \frac{1}{\pi} \tan(\pi e^{-t}) \end{aligned}$$

$$59. \frac{d^2y}{dx^2} = 2e^{-x} \Rightarrow \frac{dy}{dx} = -2e^{-x} + C; x = 0 \text{ and } \frac{dy}{dx} = 0 \Rightarrow 0 = -2e^0 + C \Rightarrow C = 2; \text{ thus } \frac{dy}{dx} = -2e^{-x} + 2$$

$$\Rightarrow y = 2e^{-x} + 2x + C_1; x = 0 \text{ and } y = 1 \Rightarrow 1 = 2e^0 + C_1 \Rightarrow C_1 = -1 \Rightarrow y = 2e^{-x} + 2x - 1 = 2(e^{-x} + x) - 1$$

$$60. \frac{d^2y}{dt^2} = 1 - e^{2t} \Rightarrow \frac{dy}{dt} = t - \frac{1}{2}e^{2t} + C; t = 1 \text{ and } \frac{dy}{dt} = 0 \Rightarrow 0 = 1 - \frac{1}{2}e^2 + C \Rightarrow C = \frac{1}{2}e^2 - 1; \text{ thus}$$

$$\frac{dy}{dt} = t - \frac{1}{2}e^{2t} + \frac{1}{2}e^2 - 1 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t + C_1; t = 1 \text{ and } y = -1 \Rightarrow -1 = \frac{1}{2} - \frac{1}{4}e^2 + \frac{1}{2}e^2 - 1 + C_1$$

$$\Rightarrow C_1 = -\frac{1}{2} - \frac{1}{4}e^2 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t - \left(\frac{1}{2} + \frac{1}{4}e^2\right)$$

$$61. f(x) = e^x - 2x \Rightarrow f'(x) = e^x - 2; f'(x) = 0 \Rightarrow e^x = 2 \Rightarrow x = \ln 2; f(0) = 1, \text{ the absolute maximum;}$$

$$f(\ln 2) = 2 - 2 \ln 2 \approx 0.613706, \text{ the absolute minimum; } f(1) = e - 2 \approx 0.71828, \text{ a relative or local maximum}$$

$$\text{since } f''(x) = e^x \text{ is always positive}$$

$$62. \text{ The function } f(x) = 2e^{\sin(x/2)} \text{ has a maximum whenever } \sin \frac{x}{2} = 1 \text{ and a minimum whenever } \sin \frac{x}{2} = -1.$$

$$\text{Therefore the maximums occur at } x = \pi + 2k(2\pi) \text{ and the minimums occur at } x = 3\pi + 2k(2\pi), \text{ where } k \text{ is any integer. The maximum is } 2e \approx 5.43656 \text{ and the minimum is } \frac{2}{e} \approx 0.73576$$

$$63. f(x) = x^2 \ln \frac{1}{x} \Rightarrow f'(x) = 2x \ln \frac{1}{x} + x^2 \left(\frac{1}{x}\right)(-x^{-2}) = 2x \ln \frac{1}{x} - x = -x(2 \ln x + 1); f'(x) = 0 \Rightarrow x = 0 \text{ or}$$

$$\ln x = -\frac{1}{2}. \text{ Since } x = 0 \text{ is not in the domain of } f, x = e^{-1/2} = \frac{1}{\sqrt{e}}. \text{ Also, } f'(x) > 0 \text{ for } 0 < x < \frac{1}{\sqrt{e}} \text{ and}$$

$$f'(x) < 0 \text{ for } x > \frac{1}{\sqrt{e}}. \text{ Therefore, } f\left(\frac{1}{\sqrt{e}}\right) = \frac{1}{e} \ln \sqrt{e} = \frac{1}{e} \ln e^{1/2} = \frac{1}{2e} \ln e = \frac{1}{2e} \text{ is the absolute maximum value}$$

$$\text{of } f \text{ assumed at } x = \frac{1}{\sqrt{e}}.$$

$$64. \int_0^{\ln 3} (e^{2x} - e^x) dx = \left[\frac{e^{2x}}{2} - e^x \right]_0^{\ln 3} = \left(\frac{e^{2 \ln 3}}{2} - e^{\ln 3} \right) - \left(\frac{e^0}{2} - e^0 \right) = \left(\frac{9}{2} - 3 \right) - \left(\frac{1}{2} - 1 \right) = \frac{8}{2} - 2 = 2$$

$$65. \text{ Let } x = \frac{r}{k} \Rightarrow k = \frac{r}{x} \text{ and as } k \rightarrow \infty, x \rightarrow 0 \Rightarrow \lim_{k \rightarrow \infty} \left(1 + \frac{r}{k}\right)^k = \lim_{x \rightarrow 0} (1 + x)^{r/x} = \lim_{x \rightarrow 0} \left((1 + x)^{1/x}\right)^r$$

$= \left(\lim_{x \rightarrow 0} (1+x)^{1/x} \right)^r$, since u^r is continuous. However, $\lim_{x \rightarrow 0} (1+x)^{1/x} = e$ (by Theorem 2), therefore,

$$\lim_{k \rightarrow \infty} \left(1 + \frac{r}{k} \right)^k = e^r.$$

66. $L = \int_0^1 \sqrt{1 + \frac{e^x}{4}} dx \Rightarrow \frac{dy}{dx} = \pm \frac{e^{x/2}}{2} \Rightarrow y = \pm e^{x/2} + C$; $y(0) = 0 \Rightarrow 0 = \pm e^0 + C \Rightarrow C = \mp 1 \Rightarrow y = e^{x/2} - 1$
 $y = -e^{x/2} + 1$

67. Note that $y = \ln x$ and $e^y = x$ are the same curve; $\int_1^a \ln x \, dx = \text{area under the curve between 1 and } a$;

$$\int_0^{\ln a} e^y \, dy = \text{area to the left of the curve. The sum of these areas is equal to the area of the rectangle}$$

$$\Rightarrow \int_1^a \ln x \, dx + \int_0^{\ln a} e^y \, dy = a \ln a.$$

68. (a) $y = e^x \Rightarrow y'' = e^x > 0$ for all $x \Rightarrow$ the graph of $y = e^x$ is always concave upward

(b) area of the trapezoid ABCD $< \int_{\ln a}^{\ln b} e^x \, dx < \text{area of the trapezoid AEFD} \Rightarrow \frac{1}{2}(AB + CD)(\ln b - \ln a)$

$$< \int_{\ln a}^{\ln b} e^x \, dx < \left(\frac{e^{\ln a} + e^{\ln b}}{2} \right) (\ln b - \ln a). \text{ Now } \frac{1}{2}(AB + CD) \text{ is the height of the midpoint}$$

$$M = e^{(\ln a + \ln b)/2} \text{ since the curve containing the points B and C is linear} \Rightarrow e^{(\ln a + \ln b)/2} (\ln b - \ln a)$$

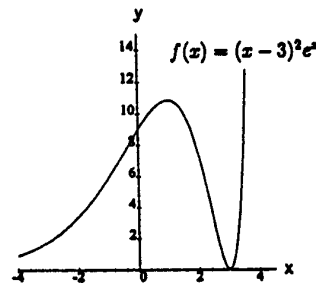
$$< \int_{\ln a}^{\ln b} e^x \, dx < \left(\frac{e^{\ln a} + e^{\ln b}}{2} \right) (\ln b - \ln a)$$

(c) $\int_{\ln a}^{\ln b} e^x \, dx = [e^x]_{\ln a}^{\ln b} = e^{\ln b} - e^{\ln a} = b - a$, so part (b) implies that

$$e^{(\ln a + \ln b)/2} (\ln b - \ln a) < b - a < \left(\frac{e^{\ln a} + e^{\ln b}}{2} \right) (\ln b - \ln a) \Rightarrow e^{(\ln a + \ln b)/2} < \frac{b - a}{\ln b - \ln a} < \frac{a + b}{2}$$

$$\Rightarrow e^{\ln(ab)/2} < \frac{b - a}{\ln b - \ln a} < \frac{a + b}{2} \Rightarrow e^{\ln(ab)^{1/2}} < \frac{b - a}{\ln b - \ln a} < \frac{a + b}{2} \Rightarrow \sqrt{ab} < \frac{b - a}{\ln b - \ln a} < \frac{a + b}{2}$$

69. $f(x) = (x-3)^2 e^x \Rightarrow f'(x) = 2(x-3)e^x + (x-3)^2 e^x$
 $= (x-3)e^x(2+x-3) = (x-1)(x-3)e^x$; thus
 $f'(x) > 0$ for $x < 1$ or $x > 3$, and $f'(x) < 0$ for
 $1 < x < 3 \Rightarrow f(1) = 4e \approx 10.87$ is a local maximum and
 $f(3) = 0$ is a local minimum. Since $f(x) \geq 0$ for all x ,
 $f(3) = 0$ is also an absolute minimum.



70. $e^{\ln x} = x$ for $x > 0$ and $\ln(e^x) = x$ for all x . My TI-92 Plus calculator gives correct results for $\ln(e^x)$, but for $x \leq 0$ it gives $e^{\ln x} = x$, which is incorrect. Try graphing $e^{\ln x}$ for $-2 \leq x \leq 2$. To be correct the line $y = x$ should not show for $x \leq 0$.

71. (a) $f(x) = e^x \Rightarrow f'(x) = e^x$; $L(x) = f(0) + f'(0)(x - 0) \Rightarrow L(x) = 1 + x$

(b) On $[0, 0.2]$, $f(x)$ and $L(x)$ are both increasing

because $f'(x) = e^x > 0$ and $L'(x) = 1 > 0$ for

$0 \leq x \leq 0.2$. Also, $f'(x) \geq L'(x)$ on the interval

$[0, 0.2]$ since $e^x \geq 1$ on the interval $E(x) = f(x) - L(x)$

is non-decreasing on $[0, 0.2]$ since $E'(x) = f'(x) - L'(x)$

≥ 0 on $[0, 0.2]$. Therefore, the largest error is

$|E(0.2)| = |f(0.2) - L(0.2)| = |e^{0.2} - 1.2| \approx 0.02141$ on $[0, 0.2]$.

(c) Since $f(x)$ is concave upward for all x , the tangent line lies below the curve $y = e^x$ for all x except at $x = 0$. Consequently, the linear approximation is never an overestimate.

72. Using Newton's Method: $f(x) = \ln(x) - 1 \Rightarrow f'(x) = \frac{1}{x} \Rightarrow x_{n+1} = x_n - \frac{\ln(x_n) - 1}{\left(\frac{1}{x_n}\right)} \Rightarrow x_{n+1} = x_n[2 - \ln(x_n)]$.

Then $x_1 = 2 \Rightarrow x_2 = 2.61370564$, $x_3 = 2.71624393$ and $x_5 = 2.71828183$. Many other methods may be used. For example, graph $y = \ln x - 1$ and determine the zero of y .

6.3 LINEAR FIRST ORDER DIFFERENTIAL EQUATIONS

1. $x \frac{dy}{dx} + y = e^x$, $x > 0$

Step 1: $\frac{dy}{dx} + \left(\frac{1}{x}\right)y = \frac{e^x}{x}$, $P(x) = \frac{1}{x}$, $Q(x) = \frac{e^x}{x}$

Step 2: $\int P(x) dx = \int \frac{1}{x} dx = \ln|x| = \ln x$, $x > 0$

Step 3: $v(x) = e^{\int P(x) dx} = e^{\ln x} = x$

Step 4: $y = \frac{1}{v(x)} \int v(x) Q(x) dx = \frac{1}{x} \int x \left(\frac{e^x}{x}\right) dx = \frac{1}{x}(e^x + C) = \frac{e^x + C}{x}$

2. $e^x \frac{dy}{dx} + 2e^x y = 1$

Step 1: $\frac{dy}{dx} + 2y = e^{-x}$, $P(x) = 2$, $Q(x) = e^{-x}$

Step 2: $\int P(x) dx = \int 2 dx = 2x$

Step 3: $v(x) = e^{\int P(x) dx} = e^{2x}$

Step 4: $y = \frac{1}{e^{2x}} \int e^{2x} \cdot e^{-x} dx = \frac{1}{e^{2x}} \int e^x dx = \frac{1}{e^{2x}}(e^x + C) = e^{-x} + Ce^{-2x}$

3. $xy' + 3y = \frac{\sin x}{x^2}, x > 0$

Step 1: $\frac{dy}{dx} + \left(\frac{3}{x}\right)y = \frac{\sin x}{x^3}, P(x) = \frac{3}{x}, Q(x) = \frac{\sin x}{x^3}$

Step 2: $\int \frac{3}{x} dx = 3 \ln |x| = \ln x^3, x > 0$

Step 3: $v(x) = e^{\ln x^3} = x^3$

Step 4: $y = \frac{1}{x^3} \int x^3 \left(\frac{\sin x}{x^3} \right) dx = \frac{1}{x^3} \int \sin x dx = \frac{1}{x^3} (-\cos x + C) = \frac{C - \cos x}{x^3}, x > 0$

4. $y' + (\tan x)y = \cos^2 x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

Step 1: $\frac{dy}{dx} + (\tan x)y = \cos^2 x, P(x) = \tan x, Q(x) = \cos^2 x$

Step 2: $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\ln |\cos x| = \ln (\cos x)^{-1}, -\frac{\pi}{2} < x < \frac{\pi}{2}$

Step 3: $v(x) = e^{\ln (\cos x)^{-1}} = (\cos x)^{-1}$

Step 4: $y = \frac{1}{(\cos x)^{-1}} \int (\cos x)^{-1} \cdot \cos^2 x dx = (\cos x) \int \cos x dx = (\cos x)(\sin x + C) = \sin x \cos x + C \cos x$

5. $x \frac{dy}{dx} + 2y = 1 - \frac{1}{x}, x > 0$

Step 1: $\frac{dy}{dx} + \left(\frac{2}{x}\right)y = \frac{1}{x} - \frac{1}{x^2}, P(x) = \frac{2}{x}, Q(x) = \frac{1}{x} - \frac{1}{x^2}$

Step 2: $\int \frac{2}{x} dx = 2 \ln |x| = \ln x^2, x > 0$

Step 3: $v(x) = e^{\ln x^2} = x^2$

Step 4: $y = \frac{1}{x^2} \int x^2 \left(\frac{1}{x} - \frac{1}{x^2} \right) dx = \frac{1}{x^2} \int (x - 1) dx = \frac{1}{x^2} \left(\frac{x^2}{2} - x + C \right) = \frac{1}{2} - \frac{1}{x} + \frac{C}{x^2}, x > 0$

6. $(1+x)y' + y = \sqrt{x}$

Step 1: $\frac{dy}{dx} + \left(\frac{1}{1+x}\right)y = \frac{\sqrt{x}}{1+x}, P(x) = \frac{1}{1+x}, Q(x) = \frac{\sqrt{x}}{1+x}$

Step 2: $\int \frac{1}{1+x} dx = \ln(1+x), \text{ since } x > 0$

Step 3: $v(x) = e^{\ln(1+x)} = 1+x$

Step 4: $y = \frac{1}{1+x} \int (1+x) \left(\frac{\sqrt{x}}{1+x} \right) dx = \frac{1}{1+x} \int \sqrt{x} dx = \left(\frac{1}{1+x} \right) \left(\frac{2}{3} x^{3/2} + C \right) = \frac{2x^{3/2}}{3(1+x)} + \frac{C}{1+x}$

7. $\frac{dy}{dx} - \frac{1}{2}y = \frac{1}{2}e^{x/2} \Rightarrow P(x) = -\frac{1}{2}, Q(x) = \frac{1}{2}e^{x/2} \Rightarrow \int P(x) dx = -\frac{1}{2}x \Rightarrow v(x) = e^{-x/2}$

$$\Rightarrow y = \frac{1}{e^{-x/2}} \int e^{-x/2} \left(\frac{1}{2} e^{x/2} \right) dx = e^{x/2} \int \frac{1}{2} dx = e^{x/2} \left(\frac{1}{2} x + C \right) = \frac{1}{2} x e^{x/2} + C e^{x/2}$$

$$8. \frac{dy}{dx} + 2y = 2xe^{-2x} \Rightarrow P(x) = 2, Q(x) = 2xe^{-2x} \Rightarrow \int P(x) dx = \int 2 dx = 2x \Rightarrow v(x) = e^{2x}$$

$$\Rightarrow y = \frac{1}{e^{2x}} \int e^{2x} (2xe^{-2x}) dx = \frac{1}{e^{2x}} \int 2x dx = e^{-2x} (x^2 + C) = x^2 e^{-2x} + C e^{-2x}$$

$$9. \frac{dy}{dx} - \left(\frac{1}{x} \right) y = 2 \ln x \Rightarrow P(x) = -\frac{1}{x}, Q(x) = 2 \ln x \Rightarrow \int P(x) dx = - \int \frac{1}{x} dx = -\ln x, x > 0$$

$$\Rightarrow v(x) = e^{-\ln x} = \frac{1}{x} \Rightarrow y = x \int \left(\frac{1}{x} \right) (2 \ln x) dx = x [(\ln x)^2 + C] = x (\ln x)^2 + Cx$$

$$10. \frac{dy}{dx} + \left(\frac{2}{x} \right) y = \frac{\cos x}{x^2}, x > 0 \Rightarrow P(x) = \frac{2}{x}, Q(x) = \frac{\cos x}{x^2} \Rightarrow \int P(x) dx = \int \frac{2}{x} dx = 2 \ln |x| = \ln x^2, x > 0$$

$$\Rightarrow v(x) = e^{\ln x^2} = x^2 \Rightarrow y = \frac{1}{x^2} \int x^2 \left(\frac{\cos x}{x^2} \right) dx = \frac{1}{x^2} \int \cos x dx = \frac{1}{x^2} (\sin x + C) = \frac{\sin x + C}{x^2}$$

$$11. \frac{ds}{dt} + \left(\frac{4}{t-1} \right) s = \frac{t+1}{(t-1)^3}, t > 1 \Rightarrow P(t) = \frac{4}{t-1}, Q(t) = \frac{t+1}{(t-1)^3} \Rightarrow \int P(t) dt = \int \frac{4}{t-1} dt$$

$$= 4 \ln |t-1| = \ln (t-1)^4 \Rightarrow v(t) = e^{\ln (t-1)^4} = (t-1)^4 \Rightarrow s = \frac{1}{(t-1)^4} \int (t-1)^4 \left[\frac{t+1}{(t-1)^3} \right] dt$$

$$= \frac{1}{(t-1)^4} \int (t^2 - 1) dt = \frac{1}{(t-1)^4} \left(\frac{t^3}{3} - t + C \right) = \frac{t^3}{3(t-1)^4} - \frac{t}{(t-1)^4} + \frac{C}{(t-1)^4}$$

$$12. (t+1) \frac{ds}{dt} + 2s = 3(t+1) + \frac{1}{(t+1)^2}, t > -1 \Rightarrow \frac{ds}{dt} + \left(\frac{2}{t+1} \right) s = 3 + \frac{1}{(t+1)^3} \Rightarrow P(t) = \frac{2}{t+1}, Q(t) = 3 + (t+1)^{-3}$$

$$\Rightarrow \int P(t) dt = \int \frac{2}{t+1} dt = 2 \ln |t+1| = \ln (t+1)^2 \Rightarrow v(t) = e^{\ln (t+1)^2} = (t+1)^2$$

$$\Rightarrow s = \frac{1}{(t+1)^2} \int (t+1)^2 [3 + (t+1)^{-3}] dt = \frac{1}{(t+1)^2} \int [3(t+1)^2 + (t+1)^{-1}] dt$$

$$= \frac{1}{(t+1)^2} [(t+1)^3 + \ln |t+1| + C] = (t+1) + (t+1)^{-2} \ln (t+1) + \frac{C}{(t+1)^2}, t > -1$$

$$13. \frac{dr}{d\theta} + (\cot \theta) r = \sec \theta, 0 < \theta < \frac{\pi}{2} \Rightarrow P(\theta) = \cot \theta, Q(\theta) = \sec \theta \Rightarrow \int P(\theta) d\theta = \int \cot \theta d\theta = \ln |\sin \theta|$$

$$\Rightarrow v(\theta) = e^{\ln |\sin \theta|} = \sin \theta \text{ because } 0 < \theta < \frac{\pi}{2} \Rightarrow r = \frac{1}{\sin \theta} \int (\sin \theta)(\sec \theta) d\theta$$

$$= \frac{1}{\sin \theta} \int \tan \theta d\theta = \frac{1}{\sin \theta} (\ln |\sec \theta| + C) = (\csc \theta) (\ln |\sec \theta| + C)$$

$$14. \tan \theta \frac{dr}{d\theta} + r = \sin^2 \theta, 0 < \theta < \frac{\pi}{2} \Rightarrow \frac{dr}{d\theta} + \frac{r}{\tan \theta} = \frac{\sin^2 \theta}{\tan \theta} \Rightarrow \frac{dr}{d\theta} + (\cot \theta) r = \sin \theta \cos \theta \Rightarrow P(\theta) = \cot \theta, Q(\theta)$$

$$= \sin \theta \cos \theta \Rightarrow \int P(\theta) d\theta = \int \cot \theta d\theta = \ln |\sin \theta| = \ln (\sin \theta) \text{ since } 0 < \theta < \frac{\pi}{2} \Rightarrow v(\theta) = e^{\ln (\sin \theta)} = \sin \theta$$

$$\Rightarrow r = \frac{1}{\sin \theta} \int (\sin \theta) (\sin \theta \cos \theta) d\theta = \frac{1}{\sin \theta} \int \sin^2 \theta \cos \theta d\theta = \left(\frac{1}{\sin \theta} \right) \left(\frac{\sin^3 \theta}{3} + C \right) = \frac{\sin^2 \theta}{3} + \frac{C}{\sin \theta}$$

$$\begin{aligned} 15. \frac{dy}{dt} + 2y &= 3 \Rightarrow P(t) = 2, Q(t) = 3 \Rightarrow \int P(t) dt = \int 2 dt = 2t \Rightarrow v(t) = e^{2t} \Rightarrow y = \frac{1}{e^{2t}} \int 3e^{2t} dt \\ &= \frac{1}{e^{2t}} \left(\frac{3}{2} e^{2t} + C \right); y(0) = 1 \Rightarrow \frac{3}{2} + C = 1 \Rightarrow C = -\frac{1}{2} \Rightarrow y = \frac{3}{2} - \frac{1}{2} e^{-2t} \end{aligned}$$

$$\begin{aligned} 16. \frac{dy}{dt} + \frac{2y}{t} &= t^2, t > 0 \Rightarrow P(t) = \frac{2}{t}, Q(t) = t^2 \Rightarrow \int P(t) dt = 2 \ln |t| \Rightarrow v(t) = e^{\ln t^2} = t^2 \Rightarrow y = \frac{1}{t^2} \int (t^2)(t^2) dt \\ &= \frac{1}{t^2} \int t^4 dt = \frac{1}{t^2} \left(\frac{t^5}{5} + C \right) = \frac{t^3}{5} + \frac{C}{t^2}; y(2) = 1 \Rightarrow \frac{8}{5} + \frac{C}{4} = 1 \Rightarrow C = -\frac{12}{5} \Rightarrow y = \frac{t^3}{5} - \frac{12}{5t^2} \end{aligned}$$

$$\begin{aligned} 17. \frac{dy}{d\theta} + \left(\frac{1}{\theta} \right) y &= \frac{\sin \theta}{\theta}, \theta > 0 \Rightarrow P(\theta) = \frac{1}{\theta}, Q(\theta) = \frac{\sin \theta}{\theta} \Rightarrow \int P(\theta) d\theta = \ln |\theta| \Rightarrow v(\theta) = e^{\ln |\theta|} = |\theta| \\ \Rightarrow y &= \frac{1}{|\theta|} \int |\theta| \left(\frac{\sin \theta}{\theta} \right) d\theta = \frac{1}{\theta} \int \theta \left(\frac{\sin \theta}{\theta} \right) d\theta \text{ for } \theta \neq 0 \Rightarrow y = \frac{1}{\theta} \int \sin \theta d\theta = \frac{1}{\theta} (-\cos \theta + C) \\ &= -\frac{1}{\theta} \cos \theta + \frac{C}{\theta}; y\left(\frac{\pi}{2}\right) = 1 \Rightarrow C = \frac{\pi}{2} \Rightarrow y = -\frac{1}{\theta} \cos \theta + \frac{\pi}{2\theta} \end{aligned}$$

$$\begin{aligned} 18. \frac{dy}{d\theta} - \left(\frac{2}{\theta} \right) y &= \theta^2 \sec \theta \tan \theta, \theta > 0 \Rightarrow P(\theta) = -\frac{2}{\theta}, Q(\theta) = \theta^2 \sec \theta \tan \theta \Rightarrow \int P(\theta) d\theta = -2 \ln |\theta| \\ \Rightarrow v(\theta) &= e^{-2 \ln |\theta|} = \theta^{-2} \Rightarrow y = \frac{1}{\theta^{-2}} \int (\theta^{-2})(\theta^2 \sec \theta \tan \theta) d\theta = \theta^2 \int \sec \theta \tan \theta d\theta = \theta^2 (\sec \theta + C) \\ &= \theta^2 \sec \theta + C\theta^2; y\left(\frac{\pi}{3}\right) = 2 \Rightarrow 2 = \left(\frac{\pi^2}{9} \right) (2) + C \left(\frac{\pi^2}{9} \right) \Rightarrow C = \frac{18}{\pi^2} - 2 \Rightarrow y = \theta^2 \sec \theta + \left(\frac{18}{\pi^2} - 2 \right) \theta^2 \end{aligned}$$

$$\begin{aligned} 19. (x+1) \frac{dy}{dx} - 2(x^2+x)y &= \frac{e^{x^2}}{x+1}, x > -1 \Rightarrow \frac{dy}{dx} - 2 \left[\frac{x(x+1)}{x+1} \right] y = \frac{e^{x^2}}{(x+1)^2} \Rightarrow \frac{dy}{dx} - 2xy = \frac{e^{x^2}}{(x+1)^2} \Rightarrow P(x) = -2x, \\ Q(x) &= \frac{e^{x^2}}{(x+1)^2} \Rightarrow \int P(x) dx = \int -2x dx = -x^2 \Rightarrow v(x) = e^{-x^2} \Rightarrow y = \frac{1}{e^{-x^2}} \int e^{-x^2} \left[\frac{e^{x^2}}{(x+1)^2} \right] dx \\ &= e^{x^2} \int \frac{1}{(x+1)^2} dx = e^{x^2} \left[\frac{(x+1)^{-1}}{-1} + C \right] = -\frac{e^{x^2}}{x+1} + Ce^{x^2}; y(0) = 5 \Rightarrow -\frac{1}{0+1} + C = 5 \Rightarrow -1 + C = 5 \\ \Rightarrow C &= 6 \Rightarrow y = 6e^{x^2} - \frac{e^{x^2}}{x+1} \end{aligned}$$

$$\begin{aligned} 20. \frac{dy}{dx} + xy &= x \Rightarrow P(x) = x, Q(x) = x \Rightarrow \int P(x) dx = \int x dx = \frac{x^2}{2} \Rightarrow v(x) = e^{x^2/2} \Rightarrow y = \frac{1}{e^{x^2/2}} \int e^{x^2/2} \cdot x dx \\ &= \frac{1}{e^{x^2/2}} \left(e^{x^2/2} + C \right) = 1 + \frac{C}{e^{x^2/2}}; y(0) = -6 \Rightarrow 1 + C = -6 \Rightarrow C = -7 \Rightarrow y = 1 - \frac{7}{e^{x^2/2}} \end{aligned}$$

$$21. \frac{dy}{dt} - ky = 0 \Rightarrow P(t) = -k, Q(t) = 0 \Rightarrow \int P(t) dt = \int -k dt = -kt \Rightarrow v(t) = e^{-kt}$$

$$\Rightarrow y = \frac{1}{e^{-kt}} \int (e^{-kt})(0) dt = e^{kt}(0 + C) = Ce^{kt}; y(0) = y_0 \Rightarrow C = y_0 \Rightarrow y = y_0 e^{kt}$$

$$22. \frac{dv}{dt} + \frac{k}{m}v = 0 \Rightarrow P(t) = \frac{k}{m}, Q(t) = 0 \Rightarrow \int P(t) dt = \int \frac{k}{m} dt = \frac{k}{m}t = \frac{kt}{m} \Rightarrow v(t) = e^{kt/m}$$

$$\Rightarrow y = \frac{1}{e^{kt/m}} \int e^{kt/m} \cdot 0 dt = \frac{C}{e^{kt/m}}; v(0) = v_0 \Rightarrow \frac{C}{e^{k(0)/m}} = v_0 \Rightarrow C = v_0 \Rightarrow v = v_0 e^{-(k/m)t}$$

$$23. x \int \frac{1}{x} dx = x(\ln|x| + C) = x \ln|x| + Cx \Rightarrow (b) \text{ is correct}$$

$$24. (a) \frac{dx}{dt} = 1000 + 0.10x = 0.01(10,000 + x) \Rightarrow dx = 0.1(10,000 + x) dt \Rightarrow \frac{dx}{x + 10,000} = 0.1 dt$$

$$\Rightarrow \int \frac{dx}{x + 10,000} = \int 0.1 dt \Rightarrow \ln|x + 10,000| = 0.1t + C_1 \Rightarrow e^{\ln|x + 10,000|} = e^{0.1t + C_1}$$

$$\Rightarrow |x + 10,000| = e^{0.1t} e^{C_1} \Rightarrow x + 10,000 = \pm C_2 e^{0.1t} \Rightarrow x(t) = -10,000 + C e^{0.1t}, \text{ where } C_2 = e^{C_1} \text{ and } C = \pm C_2. \text{ Apply the initial condition: } x(0) = 1000 = -10,000 + C e^0 \Rightarrow C = 11,000$$

$$\Rightarrow x(t) = -10,000 + 11,000 e^{0.1t}$$

$$(b) 100,000 = -10,000 + 11,000 e^{0.1t} \Rightarrow e^{0.1t} = 10 \Rightarrow t = 10 \ln(10) \approx 23.03 \approx 23 \text{ years and 11 days.}$$

25. Let $y(t)$ = the amount of salt in the container and $V(t)$ = the total volume of liquid in the tank at time t .

Then, the departure rate is $\frac{y(t)}{V(t)}$ (the outflow rate).

$$(a) \text{ Rate entering} = \frac{2 \text{ lb}}{\text{gal}} \cdot \frac{5 \text{ gal}}{\text{min}} = 10 \text{ lb/min}$$

$$(b) \text{ Volume} = V(t) = 100 \text{ gal} + (5t \text{ gal} - 4t \text{ gal}) = (100 + t) \text{ gal}$$

$$(c) \text{ The volume at time } t \text{ is } (100 + t) \text{ gal. The amount of salt in the tank at time } t \text{ is } y \text{ lbs. So the concentration at any time } t \text{ is } \frac{y}{100 + t} \text{ lbs/gal. Then, Rate leaving} = \frac{y}{100 + t} (\text{lbs/gal}) \cdot 4 (\text{gal/min}) = \frac{4y}{100 + t} \text{ lbs/min}$$

$$(d) \frac{dy}{dt} = 10 - \frac{4y}{100 + t} \Rightarrow \frac{dy}{dt} + \left(\frac{4}{100 + t}\right)y = 10 \Rightarrow P(t) = \frac{4}{100 + t}, Q(t) = 10 \Rightarrow \int P(t) dt = \int \frac{4}{100 + t} dt$$

$$= 4 \ln(100 + t) \Rightarrow v(t) = e^{4 \ln(100 + t)} = (100 + t)^4 \Rightarrow y = \frac{1}{(100 + t)^4} \int (100 + t)^4 (10 dt)$$

$$= \frac{10}{(100 + t)^4} \left(\frac{(100 + t)^5}{5} + C \right) = 2(100 + t) + \frac{C}{(100 + t)^4}; y(0) = 50 \Rightarrow 2(100 + 0) + \frac{C}{(100 + 0)^4} = 50$$

$$\Rightarrow C = -(150)(100)^4 \Rightarrow y = 2(100 + t) - \frac{(150)(100)^4}{(100 + t)^4} \Rightarrow y = 2(100 + t) - \frac{150}{\left(1 + \frac{t}{100}\right)^4}$$

$$(e) \ y(25) = 2(100 + 25) - \frac{(150)(100)^4}{(100 + 25)^4} \approx 188.56 \text{ lbs} \Rightarrow \text{concentration} = \frac{y(25)}{\text{volume}} \approx \frac{188.6}{125} \approx 1.5 \text{ lb/gal}$$

$$26. (a) \ \frac{dV}{dt} = (5 - 3) = 2 \Rightarrow V = 100 + 2t$$

The tank is full when $V = 200 = 100 + 2t \Rightarrow t = 50$ min.

(b) Let $y(t)$ be the amount of concentrate in the tank at time t .

$$\frac{dy}{dt} = \left(\frac{1}{2} \frac{\text{lb}}{\text{gal}}\right) \left(5 \frac{\text{gal}}{\text{min}}\right) - \left(\frac{y}{100 + 2t} \frac{\text{lb}}{\text{gal}}\right) \left(3 \frac{\text{gal}}{\text{min}}\right) \Rightarrow \frac{dy}{dt} = \frac{5}{2} - \frac{3}{2} \left(\frac{y}{50 + t}\right) \Rightarrow \frac{dy}{dt} + \frac{3}{2(t + 50)} y = \frac{5}{2}$$

$$Q(t) = \frac{5}{2}; \ P(t) = \frac{3}{2} \left(\frac{1}{t + 50}\right) \Rightarrow \int P(t) \, dt = \frac{3}{2} \int \frac{1}{t + 50} \, dt = \frac{3}{2} \ln(t + 50) \text{ since } t + 50 > 0$$

$$v(t) = e^{\int P(t) \, dt} = e^{\frac{3}{2} \ln(t + 50)} = (t + 50)^{3/2}$$

$$y(t) = \frac{1}{(t + 50)^{3/2}} \int \left(\frac{5}{2}\right)(t + 50)^{3/2} \, dt = (t + 50)^{-3/2} \left[(t + 50)^{5/2} + C\right] \Rightarrow y(t) = t + 50 + \frac{C}{(t + 50)^{3/2}}$$

Apply the initial condition (i.e., distilled water in the tank at $t = 0$):

$$y(0) = 0 = 50 + \frac{C}{50^{3/2}} \Rightarrow C = -50^{5/2} \Rightarrow y(t) = t + 50 - \frac{50^{5/2}}{(t + 50)^{3/2}}. \text{ When the tank is full at } t = 50,$$

$$y(50) = 100 - \frac{50^{3/2}}{100^{3/2}} \approx 83.22 \text{ pounds of concentrate.}$$

27. Let y be the amount of fertilizer in the tank at time t . Then rate entering $= 1 \frac{\text{lb}}{\text{gal}} \cdot 1 \frac{\text{gal}}{\text{min}} = 1 \frac{\text{lb}}{\text{min}}$ and the

volume in the tank at time t is $V(t) = 100 \text{ (gal)} + [1 \text{ (gal/min)} - 3 \text{ (gal/min)}]t \text{ min} = (100 - 2t) \text{ gal}$. Hence

$$\text{rate out} = \left(\frac{y}{100 - 2t}\right) 3 = \frac{3y}{100 - 2t} \text{ lbs/min} \Rightarrow \frac{dy}{dt} = \left(1 - \frac{3y}{100 - 2t}\right) \text{ lbs/min} \Rightarrow \frac{dy}{dt} + \left(\frac{3}{100 - 2t}\right)y = 1$$

$$\Rightarrow P(t) = \frac{3}{100 - 2t}, \ Q(t) = 1 \Rightarrow \int P(t) \, dt = \int \frac{3}{100 - 2t} \, dt = \frac{3 \ln(100 - 2t)}{-2} \Rightarrow v(t) = e^{(-3 \ln(100 - 2t))/2}$$

$$= (100 - 2t)^{-3/2} \Rightarrow y = \frac{1}{(100 - 2t)^{-3/2}} \int (100 - 2t)^{-3/2} \, dt = (100 - 2t)^{3/2} \left[(100 - 2t)^{-1/2} + C\right]$$

$$= (100 - 2t) + C(100 - 2t)^{3/2}; \ y(0) = 0 \Rightarrow [100 - 2(0)] + C[100 - 2(0)]^{3/2} \Rightarrow C(100)^{3/2} = -100$$

$$\Rightarrow C = -(100)^{-1/2} = -\frac{1}{10} \Rightarrow y = (100 - 2t) - \frac{(100 - 2t)^{3/2}}{10}. \text{ Let } \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -2 - \frac{\left(\frac{3}{2}\right)(100 - 2t)^{1/2}(-2)}{10}$$

$$= -2 + \frac{3\sqrt{100 - 2t}}{10} = 0 \Rightarrow 20 = 3\sqrt{100 - 2t} \Rightarrow 400 = 9(100 - 2t) \Rightarrow 400 = 900 - 18t \Rightarrow -500 = -18t$$

$\Rightarrow t \approx 27.8$ min, the time to reach the maximum. The maximum amount is then

$$y(27.8) = [100 - 2(27.8)] - \frac{[100 - 2(27.8)]^{3/2}}{10} \approx 14.8 \text{ lb}$$

28. Let $y = y(t)$ be the amount of carbon monoxide (CO) in the room at time t . The amount of CO entering the

room is $\left(\frac{4}{100} \times \frac{3}{10}\right) = \frac{12}{1000} \text{ ft}^3/\text{min}$, and the amount of CO leaving the room is $\left(\frac{y}{4500}\right)\left(\frac{3}{10}\right) = \frac{y}{15,000} \text{ ft}^3/\text{min}$.

Thus, $\frac{dy}{dt} = \frac{12}{1000} - \frac{y}{15,000} \Rightarrow \frac{dy}{dt} + \frac{1}{15,000}y = \frac{12}{1000} \Rightarrow P(t) = \frac{1}{15,000}$, $Q(t) = \frac{12}{1000} \Rightarrow v(t) = e^{t/15,000}$
 $\Rightarrow y = \frac{1}{e^{t/15,000}} \int \frac{12}{1000} e^{t/15,000} dt \Rightarrow y = e^{-t/15,000} \left(\frac{12 \cdot 15,000}{1000} e^{t/15,000} + C \right) = e^{-t/15,000} (180e^{t/15,000} + C)$;
 $y(0) = 0 \Rightarrow 0 = 1(180 + C) \Rightarrow C = -180 \Rightarrow y = 180 - 180e^{-t/15,000}$. When the concentration of CO is 0.01% in the room, the amount of CO satisfies $\frac{y}{4500} = \frac{0.01}{100} \Rightarrow y = 0.45 \text{ ft}^3$. When the room contains this amount we have $0.45 = 180 - 180e^{-t/15,000} \Rightarrow \frac{179.55}{180} = e^{-t/15,000} \Rightarrow t = -15,000 \ln\left(\frac{179.55}{180}\right) \approx 37.55 \text{ min}$.

29. Steady State = $\frac{V}{R}$ and we want $i = \frac{1}{2}\left(\frac{V}{R}\right) \Rightarrow \frac{1}{2}\left(\frac{V}{R}\right) = \frac{V}{R}(1 - e^{-Rt/L}) \Rightarrow \frac{1}{2} = 1 - e^{-Rt/L} \Rightarrow -\frac{1}{2} = -e^{-Rt/L}$
 $\Rightarrow \ln \frac{1}{2} = -\frac{Rt}{L} \Rightarrow -\frac{L}{R} \ln \frac{1}{2} = t \Rightarrow t = \frac{L}{R} \ln 2 \text{ sec}$

30. (a) $\frac{di}{dt} + \frac{R}{L}i = 0 \Rightarrow \frac{1}{i} di = -\frac{R}{L} dt \Rightarrow \ln i = -\frac{Rt}{L} + C_1 \Rightarrow i = e^{C_1} e^{-Rt/L} = Ce^{-Rt/L}$; $i(0) = I \Rightarrow I = C$
 $\Rightarrow i = Ie^{-Rt/L} \text{ amp}$

(b) $\frac{1}{2}I = Ie^{-Rt/L} \Rightarrow e^{-Rt/L} = \frac{1}{2} \Rightarrow -\frac{Rt}{L} = \ln \frac{1}{2} = -\ln 2 \Rightarrow t = \frac{L}{R} \ln 2 \text{ sec}$

(c) $t = \frac{L}{R} \Rightarrow i = Ie^{(-R/L)(L/R)} = Ie^{-1} \text{ amp}$

31. (a) $t = \frac{3L}{R} \Rightarrow i = \frac{V}{R}(1 - e^{-(R/L)(3L/R)}) = \frac{V}{R}(1 - e^{-3}) \approx 0.9502 \frac{V}{R} \text{ amp}$, or about 95% of the steady state value

(b) $t = \frac{2L}{R} \Rightarrow i = \frac{V}{R}(1 - e^{-(R/L)(2L/R)}) = \frac{V}{R}(1 - e^{-2}) \approx 0.8647 \frac{V}{R} \text{ amp}$, or about 86% of the steady state value

32. (a) $\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L} \Rightarrow P(t) = \frac{R}{L}$, $Q(t) = \frac{V}{L} \Rightarrow \int P(t) dt = \int \frac{R}{L} dt = \frac{Rt}{L} \Rightarrow v(t) = e^{Rt/L}$

$\Rightarrow i = \frac{1}{e^{Rt/L}} \int e^{Rt/L} \left(\frac{V}{L}\right) dt = \frac{1}{e^{Rt/L}} \left[\frac{L}{R} e^{Rt/L} \left(\frac{V}{L}\right) + C \right] = \frac{V}{R} + Ce^{-(R/L)t}$

(b) $i(0) = 0 \Rightarrow \frac{V}{R} + C = 0 \Rightarrow C = -\frac{V}{R} \Rightarrow i = \frac{V}{R} - \frac{V}{R} e^{-Rt/L}$

(c) $i = \frac{V}{R} \Rightarrow \frac{di}{dt} = 0 \Rightarrow \frac{di}{dt} + \frac{R}{L}i = 0 + \left(\frac{R}{L}\right)\left(\frac{V}{R}\right) = \frac{V}{L} \Rightarrow i = \frac{V}{R}$ is a solution of Eq. (11); $i = Ce^{-(R/L)t}$

$\Rightarrow \frac{di}{dt} = -\frac{RC}{L} e^{-(R/L)t} \Rightarrow \frac{di}{dt} + \frac{R}{L}i = -\frac{RC}{L} e^{-(R/L)t} + \frac{R}{L}(Ce^{-(R/L)t}) = 0 \Rightarrow i = Ce^{-(R/L)t}$ satisfies

$\frac{di}{dt} + \frac{R}{L}i = 0$

6.4 EULER'S METHOD; POPULATION MODELS

1. $y_1 = y_0 + x_0(1 - y_0) \quad dx = 0 + 1(1 - 0)(0.2) = 0.2$,

$y_2 = y_1 + x_1(1 - y_1) \quad dx = 0.2 + 1.2(1 - 0.2)(0.2) = 0.392$,

$$y_3 = y_2 + x_2(1 - y_2) \, dx = 0.392 + 1.4(1 - 0.392)(0.2) = 0.5622;$$

$$\frac{dy}{1-y} = x \, dx \Rightarrow -\ln|1-y| = \frac{x^2}{2} + C; \, x = 1, \, y = 0 \Rightarrow -\ln 1 = \frac{1}{2} + C \Rightarrow C = -\frac{1}{2} \Rightarrow \ln|1-y| = -\frac{x^2}{2} + \frac{1}{2}$$

$$\Rightarrow y = 1 - e^{(1-x^2)/2} \Rightarrow y(1.2) \approx 0.1975, \, y(1.4) \approx 0.3812, \, y(1.6) \approx 0.5416$$

$$2. \, y_1 = y_0 + \left(1 - \frac{y_0}{x_0}\right) dx = -1 + \left(1 - \frac{-1}{2}\right)(0.5) = -0.25,$$

$$y_2 = y_1 + \left(1 - \frac{y_1}{x_1}\right) dx = -0.25 + \left(1 - \frac{-0.25}{2.5}\right)(0.5) = 0.3,$$

$$y_3 = y_2 + \left(1 - \frac{y_2}{x_2}\right) dx = 0.3 + \left(1 - \frac{0.3}{3}\right)(0.5) = 0.75;$$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = 1 \Rightarrow P(x) = \frac{1}{x}, \, Q(x) = 1 \Rightarrow \int P(x) \, dx = \int \frac{1}{x} \, dx = \ln|x| = \ln x, \, x > 0 \Rightarrow v(x) = e^{\ln x} = x$$

$$\Rightarrow y = \frac{1}{x} \int x \cdot 1 \, dx = \frac{1}{x} \left(\frac{x^2}{2} + C\right); \, x = 2, \, y = -1 \Rightarrow -1 = 1 + \frac{C}{2} \Rightarrow C = -4 \Rightarrow y = \frac{x}{2} - \frac{4}{x} \Rightarrow y(2.5) = \frac{2.5}{2} - \frac{4}{2.5}$$

$$= -0.35; \, y(3.0) = \frac{3}{2} - \frac{4}{3} \approx 0.1667, \, y(3.5) = \frac{3.5}{2} - \frac{4}{3.5} = \frac{4.25}{7} \approx 0.6071$$

$$3. \, y_1 = y_0 + (2x_0y_0 + 2y_0) \, dx = 3 + [2(0)(3) + 2(3)](0.2) = 4.2,$$

$$y_2 = y_1 + (2x_1y_1 + 2y_1) \, dx = 4.2 + [2(0.2)(4.2) + 2(4.2)](0.2) = 6.216,$$

$$y_3 = y_2 + (2x_2y_2 + 2y_2) \, dx = 6.216 + [2(0.4)(6.216) + 2(6.216)](0.2) = 9.6970;$$

$$\frac{dy}{dx} = 2y(x+1) \Rightarrow \frac{dy}{y} = 2(x+1) \, dx \Rightarrow \ln|y| = (x+1)^2 + C; \, x = 0, \, y = 3 \Rightarrow \ln 3 = 1 + C \Rightarrow C = \ln 3 - 1$$

$$\Rightarrow \ln y = (x+1)^2 + \ln 3 - 1 \Rightarrow y = e^{(x+1)^2 + \ln 3 - 1} = e^{\ln 3} e^{x^2 + 2x} = 3e^{x(x+2)} \Rightarrow y(0.2) \approx 4.6581,$$

$$y(0.4) \approx 7.8351, \, y(0.6) \approx 14.2765$$

$$4. \, y_1 = y_0 + y_0^2(1 + 2x_0) \, dx = 1 + 1^2[1 + 2(-1)](0.5) = 0.5,$$

$$y_2 = y_1 + y_1^2(1 + 2x_1) \, dx = 0.5 + (0.5)^2[1 + 2(-0.5)](0.5) = 0.5,$$

$$y_3 = y_2 + y_2^2(1 + 2x_2) \, dx = 0.5 + (0.5)^2[1 + 2(0)](0.5) = 0.625;$$

$$\frac{dy}{y^2} = (1 + 2x) \, dx \Rightarrow -\frac{1}{y} = x + x^2 + C; \, x = -1, \, y = 1 \Rightarrow -1 = -1 + (-1)^2 + C \Rightarrow C = -1 \Rightarrow \frac{1}{y} = 1 - x - x^2$$

$$\Rightarrow y = \frac{1}{1 - x - x^2} \Rightarrow y(-0.5) = 0.8, \, y(0) = 1, \, y(0.5) = 4$$

$$5. \, y_1 = 1 + 1(0.2) = 1.2,$$

$$y_2 = 1.2 + (1.2)(0.2) = 1.44,$$

$$y_3 = 1.44 + (1.44)(0.2) = 1.728,$$

$$y_4 = 1.728 + (1.728)(0.2) = 2.0736,$$

$$y_5 = 2.0736 + (2.0736)(0.2) = 2.48832;$$

$$\frac{dy}{y} = dx \Rightarrow \ln y = x + C_1 \Rightarrow y = Ce^x; \, y(0) = 1 \Rightarrow 1 = Ce^0 \Rightarrow C = 1 \Rightarrow y = e^x \Rightarrow y(1) = e \approx 2.7183$$

6. $y_1 = 2 + \left(\frac{2}{1}\right)(0.2) = 2.4,$

$y_2 = 2.4 + \left(\frac{2.4}{1.2}\right)(0.2) = 2.8,$

$y_3 = 2.8 + \left(\frac{2.8}{1.4}\right)(0.2) = 3.2,$

$y_4 = 3.2 + \left(\frac{3.2}{1.6}\right)(0.2) = 3.6,$

$y_5 = 3.6 + \left(\frac{3.6}{1.8}\right)(0.2) = 4;$

$\frac{dy}{y} = \frac{dx}{x} \Rightarrow \ln y = \ln x + C \Rightarrow y = kx; y(1) = 2 \Rightarrow 2 = k \Rightarrow y = 2x \Rightarrow y(2) = 4$

7. Let $z_n = y_{n-1} + 2y_{n-1}(x_{n-1} + 1) \, dx$ and $y_n = y_{n-1} + (y_{n-1}(x_{n-1} + 1) + z_n(x_n + 1)) \, dx$ with $x_0 = 0$, $y_0 = 3$, and $dx = 0.2$. The exact solution is $y = 3e^{x(x+2)}$. Using a programmable calculator or a spreadsheet (We used a spreadsheet) gives the values in the following table.

x	z	y-approx	y-exact	Error
0	---	3	3	0
0.2	4.2	4.608	4.658122	0.050122
0.4	6.81984	7.623475	7.835089	0.211614
0.6	11.89262	13.56369	14.27646	0.712777

8. Let $z_n = y_{n-1} + x_{n-1}(1 - y_{n-1}) \, dx$ and $y_n = y_{n-1} + \left(\frac{x_{n-1}(1 - y_{n-1}) + x_n(1 - z_n)}{2}\right) \, dx$ with $x_0 = 1$, $y_0 = 0$, and $dx = 0.2$. The exact solution is $y = 1 - e^{(1-x^2)/2}$. Using a programmable calculator or a spreadsheet (We used a spreadsheet) gives the values in the following table.

x	z	y-approx	y-exact	Error
1	---	0	0	0
1.2	0.2	0.196	0.197481	0.001481
1.4	0.38896	0.378026	0.381217	0.003191
1.6	0.552178	0.536753	0.541594	0.004841

9. (a) $\frac{dP}{dt} = 0.0015P(150 - P) = \frac{0.225}{150}P(150 - P) = \frac{k}{M}P(M - P)$

Thus, $k = 0.225$ and $M = 150$, and $P = \frac{M}{1 + Ae^{-k/t}} = \frac{150}{1 + Ae^{-0.225t}}$

Initial condition: $P(0) = 6 \Rightarrow 6 = \frac{150}{1 + Ae^0} \Rightarrow 1 + A = 25 \Rightarrow A = 24$

Formula: $P = \frac{150}{1 + 24e^{-0.225t}}$

(b) $100 = \frac{150}{1 + 24e^{-0.225t}} \Rightarrow 1 + 24e^{-0.225t} = \frac{3}{2} \Rightarrow 24e^{-0.225t} = \frac{1}{2} \Rightarrow e^{-0.225t} = \frac{1}{48} \Rightarrow -0.225t = -\ln 48$

$\Rightarrow t = \frac{\ln 48}{0.225} \approx 17.21$ weeks

$125 = \frac{150}{1 + 24e^{-0.225t}} \Rightarrow 1 + 24e^{-0.225t} = \frac{6}{5} \Rightarrow 24e^{-0.225t} = \frac{1}{5} \Rightarrow e^{-0.225t} = \frac{1}{120} \Rightarrow -0.225t = -\ln 120$

$\Rightarrow t = \frac{\ln 120}{0.225} \approx 21.28$

It will take about 17.21 weeks to reach 100 guppies, and about 21.28 weeks to reach 125 guppies.

10. (a) $\frac{dP}{dt} = 0.0004P(250 - P) = \frac{0.1}{250}P(250 - P) = \frac{k}{M}P(M - P)$

Thus, $k = 0.1$ and $M = 250$, and $P = \frac{M}{1 + Ae^{-kt}} = \frac{250}{1 + Ae^{-0.1t}}$

Initial condition: $P(0) = 28$, where $t = 0$ represents the year 1970.

$$28 = \frac{250}{1 + Ae^0} \Rightarrow 28(1 + A) = 250 \Rightarrow A = \frac{250}{28} - 1 = \frac{111}{14} \approx 7.9286$$

Formula: $P(t) = \frac{250}{1 + 111e^{-0.1t}/14}$, or approximately $P(t) = \frac{250}{1 + 7.9286e^{-0.1t}}$

(b) The population $P(t)$ will round to 250 when $P(t) \geq 249.5 \Rightarrow 249.5 = \frac{250}{1 + 111e^{-0.1t}/14}$

$$\Rightarrow 249.5 \left(1 + \frac{111e^{-0.1t}}{14} \right) = 250 \Rightarrow \frac{(249.5)(111e^{-0.1t})}{14} = 0.5 \Rightarrow e^{-0.1t} = \frac{14}{55,389} \Rightarrow -0.1t = \ln \frac{14}{55,389}$$

$$\Rightarrow t = 10(\ln 55,389 - \ln 14) \approx 82.8$$

It will take about 83 years.

11. (a) Using the general solution from Example 6, part (c),

$$\frac{dy}{dt} = (0.08875 \times 10^{-7})(8 \times 10^7 - y)y \Rightarrow y(t) = \frac{M}{1 + Ae^{-rMt}} = \frac{8 \times 10^7}{1 + Ae^{-(0.08875)(8)t}} = \frac{8 \times 10^7}{1 + Ae^{-0.71t}}$$

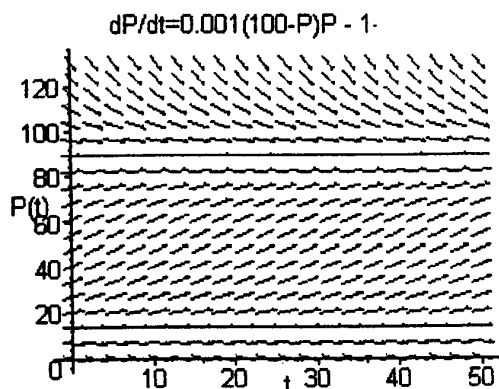
Apply the initial condition:

$$y(0) = 1.6 \times 10^7 = \frac{8 \times 10^7}{1 + A} \Rightarrow A = \frac{8}{1.6} - 1 = 4 \Rightarrow y(t) = \frac{8 \times 10^7}{1 + 4e^{-0.71t}} \approx 2.69671 \times 10^7 \text{ kg.}$$

(b) $y(t) = 4 \times 10^7 = \frac{8 \times 10^7}{1 + 4e^{-0.71t}} \Rightarrow 4e^{-0.71t} = 1 \Rightarrow t = -\frac{\ln(1/4)}{0.71} \approx 1.95253 \text{ years.}$

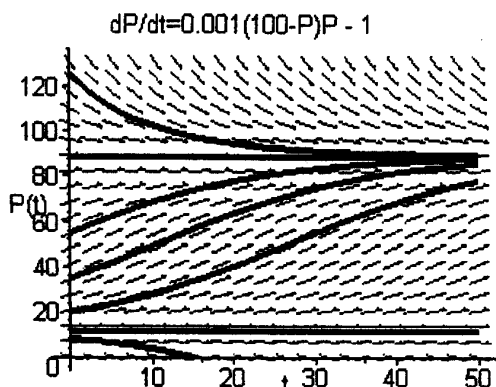
12. (a) If a part of the population leaves or is removed from the environment (e.g., a preserve or a region) each year, then c would represent the rate of reduction of the population due to this removal and/or migration. When grizzly bears become a nuisance (e.g., feeding on livestock) or threaten human safety, they are often relocated to other areas or even eliminated, but only after relocation efforts fail. In addition, bears are killed, sometimes accidentally and sometimes maliciously. For an environment that has a capacity of about 100 bears, a realistic value for c would probably be between 0 and 4.

(b)



Equilibrium solutions: $\frac{dP}{dt} = 0 = 0.001(100 - P)P - 1 \Rightarrow P^2 - 100P + 1000 = 0 \Rightarrow P_{eq} \approx 11.27$ (unstable) and $P_{eq} \approx 88.73$ (stable).

(c)



For $0 < P(0) \leq 11$, the bear population will eventually disappear, for $12 \leq P(0) \leq 88$, the population will grow to about 89, for $P(0) = 89$, the population will remain at about 89, and for $P(0) > 89$, the population will decrease to about 89 bears.

13. (a) $\frac{dy}{dt} = 1 + y \Rightarrow dy = (1 + y) dt \Rightarrow \frac{dy}{1 + y} = dt \Rightarrow \ln|1 + y| = t + C_1 \Rightarrow e^{\ln|1+y|} = e^{t+C_1} \Rightarrow |1 + y| = e^t e^{C_1}$
 $1 + y = \pm C_2 e^t \Rightarrow y = C_2 e^t - 1$, where $C_2 = e^{C_1}$ and $C = \pm C_2$. Apply the initial condition: $y(0) = 1 = C e^0 - 1 \Rightarrow C = 2 \Rightarrow y = 2e^t - 1$.
- (b) $\frac{dy}{dt} = 0.5(400 - y)y \Rightarrow dy = 0.5(400 - y)y dt \Rightarrow \frac{dy}{y(400 - y)} = 0.5 dt$. Using the partial fraction decomposition in Example 6, part (c), we obtain $\frac{1}{400} \left(\frac{1}{y} + \frac{1}{400 - y} \right) dy = 0.5 dt \Rightarrow \left(\frac{1}{y} + \frac{1}{400 - y} \right) dy = 200 dt$
 $\Rightarrow \int \left(\frac{1}{y} - \frac{1}{y - 400} \right) dy = \int 200 dt \Rightarrow \ln|y| - \ln|y - 400| = 200t + C_1 \Rightarrow \ln \left| \frac{y}{y - 400} \right| = 200t + C_1$
 $\Rightarrow e^{\ln \left| \frac{y}{y - 400} \right|} = e^{200t + C_1} = e^{200t} e^{C_1} \Rightarrow \left| \frac{y}{y - 400} \right| = C_2 e^{200t}$ (where $C_2 = e^{C_1}$) $\Rightarrow \frac{y}{y - 400} = \pm C_2 e^{200t}$
 $\Rightarrow \frac{y}{y - 400} = C e^{200t}$ (where $C = \pm C_2$) $\Rightarrow y = C e^{200t} y - 400 C e^{200t} \Rightarrow (1 - C e^{200t}) y = -400 C e^{200t}$
 $\Rightarrow y = \frac{400 C e^{200t}}{C e^{200t} - 1} \Rightarrow y = \frac{400}{1 - \frac{1}{C} e^{-200t}} = \frac{400}{1 + A e^{-200t}}$, where $A = -\frac{1}{C}$. Apply the initial condition:
 $y(0) = 2 = \frac{400}{1 + A e^0} \Rightarrow A = 199 \Rightarrow y(t) = \frac{400}{1 + 199 e^{-200t}}$.

14. $\frac{dP}{dt} = r(M - P)P \Rightarrow dP = r(M - P)P dt \Rightarrow \frac{dP}{P(M - P)} = r dt$. Using the partial fraction decomposition in Example 6, part (c), we obtain $\frac{1}{M} \left(\frac{1}{P} + \frac{1}{M - P} \right) dP = r dt \Rightarrow \left(\frac{1}{P} + \frac{1}{M - P} \right) dP = rM dt \Rightarrow \int \left(\frac{1}{P} - \frac{1}{P - M} \right) dP = \int rM dt$
 $\Rightarrow \ln|P| - \ln|P - M| = (rM)t + C_1 \Rightarrow \ln \left| \frac{P}{P - M} \right| = (rM)t + C_1 \Rightarrow e^{\ln \left| \frac{P}{P - M} \right|} = e^{(rM)t + C_1}$

$$\begin{aligned}
&= e^{(rM)t} e^{C_1} \Rightarrow \left| \frac{P}{P-M} \right| = C_2 e^{(rM)t} \text{ (where } C_2 = e^{C_1}) \Rightarrow \frac{P}{P-M} = \pm C_2 e^{(rM)t} \Rightarrow \frac{P}{P-M} = C e^{(rM)t} \text{ (where } \\
&C = \pm C_2) \Rightarrow P = C e^{(rM)t} P - M C e^{(rM)t} \Rightarrow (1 - C e^{(rM)t}) P = -M C e^{(rM)t} \Rightarrow P = \frac{M C e^{(rM)t}}{C e^{(rM)t} - 1} \\
&\Rightarrow P = \frac{M}{1 - \frac{1}{C} e^{-(rM)t}} = \frac{M}{1 + A e^{-(rM)t}}, \text{ where } A = -\frac{1}{C}.
\end{aligned}$$

$$15. (a) \frac{dP}{dt} = kP^2 \Rightarrow \int P^{-2} dP = \int k dt \Rightarrow -P^{-1} = kt + C \Rightarrow P = \frac{-1}{kt + C}$$

$$\text{Initial condition: } P(0) = P_0 \Rightarrow P_0 = -\frac{1}{C} \Rightarrow C = -\frac{1}{P_0}$$

$$\text{Solution: } P = -\frac{1}{kt - (1/P_0)} = \frac{P_0}{1 - kP_0 t}$$

$$(b) \text{ There is a vertical asymptote at } t = \frac{1}{kP_0}$$

$$16. (a) \frac{dP}{dt} = r(M - P)(P - m) \Rightarrow \frac{dP}{dt} = r(1200 - P)(P - 100) \Rightarrow \frac{1}{(1200 - P)(P - 100)} \frac{dP}{dt} = r$$

$$\Rightarrow \frac{1100}{(1200 - P)(P - 100)} \frac{dP}{dt} = 1100r \Rightarrow \frac{(P - 100) + (1200 - P)}{(1200 - P)(P - 100)} \frac{dP}{dt} = 1100r$$

$$\Rightarrow \left(\frac{1}{1200 - P} + \frac{1}{P - 100} \right) \frac{dP}{dt} = 1100r$$

$$(b) \left(\frac{1}{1200 - P} + \frac{1}{P - 100} \right) dP = 1100r dt \Rightarrow \int \left(\frac{1}{1200 - P} + \frac{1}{P - 100} \right) dP = \int 1100r dt$$

$$\Rightarrow \ln(1200 - P) + \ln(P - 100) = 1100rt + C_1 \Rightarrow \left| \frac{P - 100}{1200 - P} \right| = 1100rt + C_1 \Rightarrow \frac{P - 100}{1200 - P}$$

$$= \pm e^{C_1} e^{1100rt} \Rightarrow \frac{P - 100}{1200 - P} = C e^{1100rt} \text{ where } C = \pm e^{C_1} \Rightarrow P - 100 = 1200 C e^{1100rt} - C P e^{1100rt}$$

$$\Rightarrow P(1 + C e^{1100rt}) = 1200 C e^{1100rt} + 100 \Rightarrow P = \frac{1200 C e^{1100rt} + 100}{C e^{1100rt} + 1} \Rightarrow \frac{1200 + \frac{100}{C} e^{-1100rt}}{1 + \frac{1}{C} e^{-1100rt}}$$

$$\Rightarrow P = \frac{1200 + 100 A e^{-1100rt}}{1 + A e^{-1100rt}} \text{ where } A = \frac{1}{C}.$$

$$\text{Apply the initial condition: } 300 = \frac{1200 + 100A}{1 + A} = 300 + 300A = 1200 + 100A \Rightarrow A = \frac{9}{2}$$

$$\Rightarrow P = \frac{2400 + 900 e^{-1100rt}}{2 + 9 e^{-1100rt}}. \text{ (Note that } P \rightarrow 1200 \text{ as } t \rightarrow \infty.)$$

$$(c) \frac{dP}{dt} = r(M - P)(P - m) \Rightarrow \frac{dP}{dt} = r(M - P)(P - m) \Rightarrow \frac{1}{(M - P)(P - m)} \frac{dP}{dt} = r \Rightarrow \frac{M - m}{(M - P)(P - m)} \frac{dP}{dt}$$

$$= (M - m)r \Rightarrow \frac{(P - m) + (M - P)}{(M - P)(P - m)} \frac{dP}{dt} = (M - m)r \Rightarrow \left(\frac{1}{M - P} + \frac{1}{P - m} \right) \frac{dP}{dt} = (M - m)r$$

$$\Rightarrow \int \left(\frac{1}{M - P} + \frac{1}{P - m} \right) dP = \int (M - m)r dt \Rightarrow -\ln(M - P) + \ln(P - m) = (M - m)rt + C_1$$

$$\Rightarrow \ln \left| \frac{P-m}{M-P} \right| = (M-m)rt + C_1 \Rightarrow \frac{P-m}{M-P} = \pm e^{C_1} e^{(M-m)rt} \Rightarrow \frac{P-m}{M-P} = C e^{(M-m)rt} \text{ where } C = \pm e^{C_1}$$

$$\Rightarrow P-m = M C e^{(M-m)rt} - C P e^{(M-m)rt} \Rightarrow P(1 + C e^{(M-m)rt}) = M C e^{(M-m)rt} + m \Rightarrow P = \frac{M C e^{(M-m)rt} + m}{C e^{(M-m)rt} + 1}$$

$$\Rightarrow P = \frac{M + \frac{m}{C} e^{-(M-m)rt}}{1 + \frac{1}{C} e^{-(M-m)rt}} \Rightarrow P = \frac{M + m A e^{-(M-m)rt}}{1 + A e^{-(M-m)rt}} \text{ where } A = \frac{1}{C}.$$

Apply the initial condition $P(0) = P_0$:

$$P_0 = \frac{M + m A}{1 + A} \Rightarrow P_0 + P_0 A = M + m A \Rightarrow A = \frac{M - P_0}{P_0 - m} \Rightarrow P = \frac{M(P_0 - m) + m(M - P_0)e^{-(M-m)rt}}{(P_0 - m) + (M - P_0)e^{-(M-m)rt}}$$

(Note that $P \rightarrow M$ as $t \rightarrow \infty$ provided $P_0 > m$.)

$$17. \frac{dy}{dx} = 2xe^{x^2}, y(0) = 2 \Rightarrow y_{n+1} = y_n + 2x_n e^{x_n^2} dx = y_n + 2x_n e^{x_n^2}(0.1) = y_n + 0.2x_n e^{x_n^2}$$

On a TI-92 Plus calculator home screen, type the following commands:

2 STO> y: 0 STO> x:y (enter)

$y+0.2*x*e^{(x^2)}$ STO> y: $x+0.1$ STO> x: y (enter, 10 times)

The last value displayed gives $y_{\text{Euler}}(1) \approx 3.45835$

The exact solution: $dy = 2xe^{x^2} dx \Rightarrow y = e^{x^2} + C$; $y(0) = 2 = e^0 + C \Rightarrow C = 1 \Rightarrow y = 1 + e^{x^2}$

$\Rightarrow y_{\text{exact}}(1) = 1 + e \approx 3.71828$

$$18. \frac{dy}{dx} = y + e^x - 2, y(0) = 2 \Rightarrow y_{n+1} = y_n + (y_n + e^{x_n} - 2) dx = y_n + 0.5(y_n + e^{x_n} - 2)$$

On a TI-92 Plus calculator home screen, type the following commands:

2 STO> y: 0 STO> x:y (enter)

$y+0.5*(y+e^x-2)$ STO> y: $x+0.5$ STO> x: y (enter, 4 times)

The last value displayed gives $y_{\text{Euler}}(2) \approx 9.82187$

The exact solution: $\frac{dy}{dx} - y = e^x - 2 \Rightarrow P(x) = -1, Q(x) = e^x - 2 \Rightarrow \int P(x) dx = -x \Rightarrow v(x) = e^{-x}$

$\Rightarrow y = \frac{1}{e^{-x}} \int e^{-x}(e^x - 2) dx = e^x(x + 2e^{-x} + C)$; $y(0) = 2 \Rightarrow 2 = 2 + C \Rightarrow C = 0$

$\Rightarrow y = xe^x + 2 \Rightarrow y_{\text{exact}}(2) = 2e^2 + 2 \approx 16.7781$

$$19. y_1 = -1 + \left[\frac{(-1)^2}{\sqrt{1}} \right] (0.5) = -0.5,$$

$$y_2 = -0.5 + \left[\frac{(-0.5)^2}{\sqrt{1.5}} \right] (0.5) = -0.39794,$$

$$y_3 = -0.39794 + \left[\frac{(-0.39794)^2}{\sqrt{2}} \right] (0.5) = -0.34195,$$

$$y_4 = -0.34195 + \left[\frac{(-0.34195)^2}{\sqrt{2.5}} \right] (0.5) = -0.30497,$$

$$y_5 = -0.27812, y_6 = -0.25745, y_7 = -0.24088, y_8 = -0.2272;$$

$$\frac{dy}{y^2} = \frac{dx}{\sqrt{x}} \Rightarrow -\frac{1}{y} = 2\sqrt{x} + C; y(1) = -1 \Rightarrow 1 = 2 + C \Rightarrow C = -1 \Rightarrow y = \frac{1}{1-2\sqrt{x}} \Rightarrow y(5) = \frac{1}{1-2\sqrt{5}} \approx -0.2880$$

$$20. y_1 = 1 + (1 - e^0)\left(\frac{1}{3}\right) = 1,$$

$$y_2 = 1 + (1 - e^{2/3})\left(\frac{1}{3}\right) = 0.68409,$$

$$y_3 = 0.68409 + (0.68409 - e^{4/3})\left(\frac{1}{3}\right) = -0.35244,$$

$$y_4 = -0.35244 + (-0.35244 - e^{6/3})\left(\frac{1}{3}\right) = -2.93294,$$

$$y_5 = -2.93294 + (-2.93294 - e^{8/3})\left(\frac{1}{3}\right) = -8.70789,$$

$$y_6 = -8.70789 + (-8.70789 - e^{10/3})\left(\frac{1}{3}\right) = -20.95439;$$

$$y' - y = -e^{2x} \Rightarrow P(x) = -1, Q(x) = -e^{2x} \Rightarrow \int P(x) dx = -x \Rightarrow v(x) = e^{-x} \Rightarrow y = \frac{1}{e^{-x}} \int e^{-x}(-e^{2x}) dx$$

$$= e^x(-e^x + C); y(0) = 1 \Rightarrow 1 = -1 + C \Rightarrow C = 2 \Rightarrow y = -e^{2x} + 2e^x \Rightarrow y(2) = -e^4 + 2e^2 \approx -39.8200$$

$$21. (a) \frac{dy}{dx} = 2y^2(x-1) \Rightarrow \frac{dy}{y^2} = 2(x-1) dx \Rightarrow \int y^{-2} dy = \int (2x-2) dx \Rightarrow -y^{-1} = x^2 - 2x + C$$

$$\text{Initial value: } y(2) = -\frac{1}{2} \Rightarrow 2 = 2^2 - 2(2) + C \Rightarrow C = 2$$

$$\text{Solution: } -y^{-1} = x^2 - 2x + 2 \text{ or } y = -\frac{1}{x^2 - 2x + 2}$$

$$y(3) = -\frac{1}{3^2 - 2(3) + 2} = -\frac{1}{5} = -0.2$$

(b) To find the approximation, set $y_1 = 2y^2(x-1)$ and use EULERT with initial values $x = 2$ and $y = -\frac{1}{2}$ and step size 0.2 for 5 points. This gives $y(3) \approx -0.1851$; error ≈ 0.0149 .

(c) Use step size 0.1 for 10 points. This gives $y(3) \approx -0.1929$; error ≈ 0.0071 .

(d) Use step size 0.05 for 20 points. This gives $y(3) \approx -0.1965$; error ≈ 0.0035 .

$$22. (a) \frac{dy}{dx} = y - 1 \Rightarrow \int \frac{dy}{y-1} = \int dx \Rightarrow \ln |y-1| = x + C \Rightarrow |y-1| = e^{x+C} \Rightarrow y-1 = \pm e^C e^x$$

$$\Rightarrow y = Ae^x + 1$$

$$\text{Initial condition: } y(0) = 3 \Rightarrow 3 = Ae^0 + 1 \Rightarrow A = 2$$

$$\text{Solution: } y = 2e^x + 1$$

$$y(1) = 2e + 1 \approx 6.4366$$

(b) To find the approximation, set $y_1 = y - 1$ and use a graphing calculator or CAS with initial values $x = 0$ and $y = 3$ and step size 0.2 for 5 points. This gives $y(1) \approx 5.9766$; error ≈ 0.4599 .

(c) Use step size 0.1 for 10 points. This gives $y(1) \approx 6.1875$; error ≈ 0.2491 .

(d) Use step size 0.05 for 20 points. This gives $y(1) \approx 6.3066$; error ≈ 0.1300 .

23. The exact solution is $y = \frac{-1}{x^2 - 2x + 2}$, so $y(3) = -0.2$. To find the approximation, let

$$z_n = y_{n-1} + 2y_{n-1}^2(x_n - 1) \, dx \text{ and } y_n = y_{n-1} + (y_{n-1}^2(x_{n-1} - 1) + z_n^2(x_n^2 - 1)) \, dx \text{ with initial values } x_0 = 2$$

and $y_0 = -\frac{1}{2}$. Use a spreadsheet, graphing calculator, or CAS as indicated in parts (a) through (d).

- (a) Use $dx = 0.2$ with 5 steps to obtain $y(3) \approx -0.2024 \Rightarrow \text{error} \approx 0.0024$.
- (b) Use $dx = 0.1$ with 10 steps to obtain $y(3) \approx -0.2005 \Rightarrow \text{error} \approx 0.0005$.
- (c) Use $dx = 0.05$ with 20 steps to obtain $y(3) \approx -0.2001 \Rightarrow \text{error} \approx 0.0001$.
- (d) Each time the step size is cut in half, the error is reduced to approximately one-fourth of what it was for the larger step size.

24. The exact solution is $y = 2e^x + 1$, so $y(1) = 2e^1 + 1 \approx 6.4366$. To find the approximate solution let

$$z_n = y_{n-1} + (y_{n-1} - 1) \, dx \text{ and } y_n = y_{n-1} + \left(\frac{y_{n-1} + z_n - 2}{2} \right) dx \text{ with initial value } y_n = 3. \text{ Use a spreadsheet, graphing calculator, or CAS as indicated in parts (a) through (d).}$$

- (a) Use $dx = 0.2$ with 5 steps to obtain $y(1) \approx 6.4054 \Rightarrow \text{error} \approx 0.0311$.
- (b) Use $dx = 0.1$ with 10 steps to obtain $y(1) \approx 6.4282 \Rightarrow \text{error} \approx 0.0084$.
- (c) Use $dx = 0.05$ with 20 steps to obtain $y(1) \approx 6.4344 \Rightarrow \text{error} \approx 0.0022$.
- (d) Each time the step size is cut in half, the error is reduced to approximately one-fourth of what it was for the larger step size.

25-30. Example CAS commands:

Maple:

```
with(plots): with(DEtools):
a:=-4; b:=4;
eq:= D(y)(x) = x + y;
plot1:= dfieldplot(eq,[x,y], x=a..b, y=-4..4, scaling=CONSTRAINED);
display({plot1});
gen_sol:= dsolve({eq},y(x));
tograph:= {seq(subs(_C1=i, gen_sol), i = {-1,0,1,3,9})};
plot2:=implicitplot(tograph, x=a..b, y=-4..4, scaling=CONSTRAINED);
display({plot1,plot2}, title = 'Direction Field and Solution Curves');
eulerapprox:= proc(f,x0,y0,n) local i,j,h;
    x(0):= evalf(x0);
    y(0):= evalf(y0);
    h:= (b-a)/n;
    for i from 1 to n do
        y(i):= evalf(y(i-1) + h*f(x(i-1),y(i-1))); x(i):= x(i-1) + h od;
    [[x(j),y(j)] $j=0..n];
end;
rhs(eq);
f:= unapply(%,(x,y));
eulerapprox(f,0,-7/10,4);
plot3:= plot(%, style=LINE,scaling=CONSTRAINED, title='Euler Approximation');
display({plot3});
y(0):= 'y(0)';
partsol:= dsolve({eq, y(0)=-7/10}, y(x));
plot4:= implicitplot(partsol,x=-1..8,y=-3..40,scaling=CONSTRAINED);
display({plot3, plot4}, title='Actual Solution & Euler Approximation');
```

Mathematica:

Need package for plotting vector fields:

```
<< "Graphics`PlotField`"
```

Also load package to improve solving of ODE's:

```
<< Calculus`DSolve`
```

```
SetOptions[PlotVectorField, PlotPoints -> 6];
```

```
Clear[x,y,yp,h]
```

Note: here we define "eulerstep" to find the next Euler point, given the current one, assuming that the variables "a" (initial point), "b" (final point), and "n" (# of steps) have been defined, along with the function "yp[x,y]" (which specifies the derivative). Then the whole Euler solution is given as a list of points {x,y} by:

```
NestList[ eulerstep, N[{a, ya}], n ]
```

where "ya" is the initial value.

```
eulerstep[{x_,y_}] := {x+h, y+h*N[yp[x,y]] }
```

```
h := N[(b-a)/n]
```

```
yp[x_,y_] := x+y
```

```
{a,b} = {0,1}; ya = 1;
```

```
{xmin,xmax} = {-4,4}; {ymin,ymax} = {-4,4};
```

```
p1 = PlotVectorField[{1,yp[x,y]},{x,xmin,xmax},{y,ymin,ymax},
```

```
ScaleFunction -> (1&)]
```

```
ode = y'[x] == yp[x,y[x]]
```

```
DSolve[ode, y[x], x]
```

```
gensol = y[x] /. First[%]
```

```
sols = Map[ (gensol /. C[1] -> #)&, {-2,-1,0,1,2} ]
```

```
p2 = Plot[ Evaluate[sols], {x,xmin,xmax} ]
```

```
Show[ {p1, p2}, PlotRange -> {Automatic,{ymin,ymax}} ]
```

```
DSolve[ {ode, y[a] == ya}, y[x], x ]
```

```
partsol = y[x] /. First[%]
```

```
p3 = Plot[ partsol, {x,a,b} ]
```

```
n = 10;
```

```
approx1 = NestList[ eulerstep, N[{a, ya}], n ];
```

```
p4 = ListPlot[ approx1, PlotJoined -> True ]
```

```
Show[{p4,p3}]
```

Here's an alternate approach to plotting the two solutions (simpler but less obvious):

```
Show[p3, Epilog -> {Line[approx]} ]
```

```
n = 25;
```

```
approx2 = NestList[ eulerstep, N[{a, ya}], n ];
```

```
p4 = ListPlot[ approx2, PlotJoined -> True ]
```

```
Show[{p4,p3}]
```

```
n = 50;
```

```
approx3 = NestList[ eulerstep, N[{a, ya}], n ];
```

```
p4 = ListPlot[ approx3, PlotJoined -> True ]
```

```
Show[{p4,p3}]
```

```
n = 100;
```

```
approx4 = NestList[ eulerstep, N[{a, ya}], n ];
```

```
p4 = ListPlot[ approx4, PlotJoined -> True ]
```

```
Show[{p4,p3}]
```

```
yb = partsol /. x -> b // N
```

```
err1 = Last[approx1][[2]] - yb
```

```
percent1 = err1/yb * 100
```

```

err2 = Last[approx2][[2]] - yb
percent2 = err2/yb * 100
err3 = Last[approx3][[2]] - yb
percent3 = err3/yb * 100
err4 = Last[approx4][[2]] - yb
percent4 = err4/yb * 100

```

31. Example CAS commands:

Maple:

```

with(plots): with(DEtools):
eq:= D(y)(x) = f - y;
eq1:= subs(f = 2*x, eq);
eq2:= subs(f=sin(2*x), eq);
eq3:= subs(f=3*exp(x/2), eq);
eq4:= subs(f=2*exp(-x/2)*cos(2*x), eq);
partsol1:= dsolve({eq1,y(0)=0}, y(x));
plot1:= implicitplot(partsol1, x=-2..6, y=-1..10, scaling=CONSTRAINED):
display(plot1);
partsol2:= dsolve({eq2, y(0)=0}, y(x));
plot2:= implicitplot(partsol2, x=-2..6, y=-1..4, scaling=CONSTRAINED):
display(plot2);
partsol3:= dsolve({eq3, y(0)=0}, y(x));
plot3:= implicitplot(partsol3, x=-2..6, y=-2..10, scaling=CONSTRAINED):
display(plot3);
partsol4:= dsolve({eq4, y(0)=0}, y(x));
plot4:= implicitplot(partsol4, x=-2..6, y=-3..2, scaling=CONSTRAINED):
display(plot4);
display({plot1,plot2,plot3,plot4});

```

Mathematica:

```

Clear[x,y,f]
ode = y'[x] + y[x] == f[x]
a = 0; ya = 0;
{xmin,xmax} = {-2,6};
f[x_] = 2x
DSolve[ {ode, y[a] == ya}, y[x], x ]
sol1 = y[x] /. First[%]
Plot[ sol1, {x,xmin,xmax} ]
f[x_] = Sin[2x]
DSolve[ {ode, y[a] == ya}, y[x], x ]
sol2 = y[x] /. First[%]
Plot[ sol2, {x,xmin,xmax} ]
f[x_] = 3 Exp[x/2]
DSolve[ {ode, y[a] == ya}, y[x], x ]
sol3 = y[x] /. First[%]
Plot[ sol3, {x,xmin,xmax} ]
f[x_] = 2 Exp[-x/2] Cos[2x]
DSolve[ {ode, y[a] == ya}, y[x], x ]
sol4 = y[x] /. First[%]
Plot[ sol4, {x,xmin,xmax} ]
Plot[ {sol1, sol2, sol3, sol4}, {x,xmin,xmax} ]

```

32. Example CAS commands:

Maple:

```

with(plots): with(DEtools):
a:=-3; b:=3;
eq:=D(y)(x)=(3*x^2+4*x+2)/(2*(y-1));
plot1:=dfieldplot(eq,[x,y],x=a..b,y=a..b,scaling=CONSTRAINED);
display({plot1});
right:=int(numer(rhs(eq)),x);
left:=int(denom(rhs(eq)),y);
sol:=left = right + C;
tograph:={seq(subs(C=i,sol),i={-6,-4,-2,0,2,4,6})};
plot2:=implicitplot(tograph,x=a..b,y=a..b, scaling=CONSTRAINED);
display(plot2);
DEplot(eq,y(x),x=a..b,{[0,-1]},y=a..b);

```

Mathematica:

```

yp[x_,y_] := (3x^2 + 4x + 2) / (2(y-1))
a = 0; ya = -1;
{xmin,xmax} = {-3,3}; {ymin,ymax} = {-3,3};
p1 = PlotVectorField[{1,yp[x,y]}, {x,xmin,xmax}, {y,ymin,ymax},
  ScaleFunction -> (1&)]
impeqn =
  Integrate[Denominator[yp[x,y]],y] ==
  Integrate[Numerator[yp[x,y]],x] + C[1]
<< Graphics`ImplicitPlot`
eqns = Map[ (impeqn /. C[1] -> #)&,
  {-6,-4,-2,0,2,4,6} ];
p2 = ImplicitPlot[ Evaluate[eqns], {x,xmin,xmax} ]
Show[ {p1, p2} ]
impeqn /. {x -> 0, y -> -1}
Solve[%,C[1]]
parteqn = impeqn /. First[%]
ImplicitPlot[ Evaluate[parteqn], {x,xmin,xmax} ]

```

6.5 HYPERBOLIC FUNCTIONS

$$1. \sinh x = -\frac{3}{4} \Rightarrow \cosh x = \sqrt{1 + \sinh^2 x} = \sqrt{1 + \left(-\frac{3}{4}\right)^2} = \sqrt{1 + \frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}, \tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(-\frac{3}{4}\right)}{\left(\frac{5}{4}\right)} = -\frac{3}{5},$$

$$\coth x = \frac{1}{\tanh x} = -\frac{5}{3}, \operatorname{sech} x = \frac{1}{\cosh x} = \frac{4}{5}, \text{ and } \operatorname{csch} x = \frac{1}{\sinh x} = -\frac{4}{3}$$

$$2. \sinh x = \frac{4}{3} \Rightarrow \cosh x = \sqrt{1 + \sinh^2 x} = \sqrt{1 + \frac{16}{9}} = \sqrt{\frac{25}{9}} = \frac{5}{3}, \tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{4}{3}\right)}{\left(\frac{5}{3}\right)} = \frac{4}{5}, \coth x = \frac{1}{\tanh x} = \frac{5}{4},$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{3}{5}, \text{ and } \operatorname{csch} x = \frac{1}{\sinh x} = \frac{3}{4}$$

$$3. \cosh x = \frac{17}{15}, x > 0 \Rightarrow \sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\left(\frac{17}{15}\right)^2 - 1} = \sqrt{\frac{289}{225} - 1} = \sqrt{\frac{64}{225}} = \frac{8}{15}, \tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{8}{15}\right)}{\left(\frac{17}{15}\right)}$$

$$= \frac{8}{17}, \coth x = \frac{1}{\tanh x} = \frac{17}{8}, \operatorname{sech} x = \frac{1}{\cosh x} = \frac{15}{17}, \text{ and } \operatorname{csch} x = \frac{1}{\sinh x} = \frac{15}{8}$$

$$4. \cosh x = \frac{13}{5}, x > 0 \Rightarrow \sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\frac{169}{25} - 1} = \sqrt{\frac{144}{25}} = \frac{12}{5}, \tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{12}{5}\right)}{\left(\frac{13}{5}\right)} = \frac{12}{13},$$

$$\coth x = \frac{1}{\tanh x} = \frac{13}{12}, \operatorname{sech} x = \frac{1}{\cosh x} = \frac{5}{13}, \text{ and } \operatorname{csch} x = \frac{1}{\sinh x} = \frac{5}{12}$$

$$5. 2 \cosh(\ln x) = 2 \left(\frac{e^{\ln x} + e^{-\ln x}}{2} \right) = e^{\ln x} + \frac{1}{e^{\ln x}} = x + \frac{1}{x}$$

$$6. \sinh(2 \ln x) = \frac{e^{2 \ln x} - e^{-2 \ln x}}{2} = \frac{e^{\ln x^2} - e^{\ln x^{-2}}}{2} = \frac{\left(x^2 - \frac{1}{x^2}\right)}{2} = \frac{x^4 - 1}{2x^2}$$

$$7. \cosh 5x + \sinh 5x = \frac{e^{5x} + e^{-5x}}{2} + \frac{e^{5x} - e^{-5x}}{2} = e^{5x}$$

$$8. \cosh 3x - \sinh 3x = \frac{e^{3x} + e^{-3x}}{2} - \frac{e^{3x} - e^{-3x}}{2} = e^{-3x}$$

$$9. (\sinh x + \cosh x)^4 = \left(\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} \right)^4 = (e^x)^4 = e^{4x}$$

$$10. \ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x) = \ln(\cosh^2 x - \sinh^2 x) = \ln 1 = 0$$

$$11. (a) \sinh 2x = \sinh(x + x) = \sinh x \cosh x + \cosh x \sinh x = 2 \sinh x \cosh x$$

$$(b) \cosh 2x = \cosh(x + x) = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x$$

$$12. \cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2 = \frac{1}{4} [(e^x + e^{-x}) + (e^x - e^{-x})] [(e^x + e^{-x}) - (e^x - e^{-x})]$$

$$= \frac{1}{4} (2e^x)(2e^{-x}) = \frac{1}{4} (4e^0) = \frac{1}{4} (4) = 1$$

$$13. y = 6 \sinh \frac{x}{3} \Rightarrow \frac{dy}{dx} = 6 \left(\cosh \frac{x}{3} \right) \left(\frac{1}{3} \right) = 2 \cosh \frac{x}{3}$$

$$14. y = \frac{1}{2} \sinh(2x + 1) \Rightarrow \frac{dy}{dx} = \frac{1}{2} [\cosh(2x + 1)](2) = \cosh(2x + 1)$$

$$15. y = 2\sqrt{t} \tanh \sqrt{t} = 2t^{1/2} \tanh t^{1/2} \Rightarrow \frac{dy}{dt} = [\operatorname{sech}^2(t^{1/2})] \left(\frac{1}{2} t^{-1/2} \right) (2t^{1/2}) + (\tanh t^{1/2}) (t^{-1/2})$$

$$= \operatorname{sech}^2 \sqrt{t} + \frac{\tanh \sqrt{t}}{\sqrt{t}}$$

$$16. y = t^2 \tanh \frac{1}{t} = t^2 \tanh t^{-1} \Rightarrow \frac{dy}{dt} = [\operatorname{sech}^2(t^{-1})] (-t^{-2}) (t^2) + (2t) (\tanh t^{-1}) = -\operatorname{sech}^2 \frac{1}{t} + 2t \tanh \frac{1}{t}$$

$$17. y = \ln(\sinh z) \Rightarrow \frac{dy}{dz} = \frac{\cosh z}{\sinh z} = \coth z$$

$$18. y = \ln(\cosh z) \Rightarrow \frac{dy}{dz} = \frac{\sinh z}{\cosh z} = \tanh z$$

$$19. y = (\operatorname{sech} \theta)(1 - \ln \operatorname{sech} \theta) \Rightarrow \frac{dy}{d\theta} = \left(-\frac{\operatorname{sech} \theta \tanh \theta}{\operatorname{sech} \theta} \right) (\operatorname{sech} \theta) + (-\operatorname{sech} \theta \tanh \theta)(1 - \ln \operatorname{sech} \theta)$$

$$= \operatorname{sech} \theta \tanh \theta - (\operatorname{sech} \theta \tanh \theta)(1 - \ln \operatorname{sech} \theta) = (\operatorname{sech} \theta \tanh \theta)[1 - (1 - \ln \operatorname{sech} \theta)]$$

$$= (\operatorname{sech} \theta \tanh \theta)(\ln \operatorname{sech} \theta)$$

$$\begin{aligned}
 20. \quad y &= (\operatorname{csch} \theta)(1 - \ln \operatorname{csch} \theta) \Rightarrow \frac{dy}{d\theta} = (\operatorname{csch} \theta) \left(-\frac{-\operatorname{csch} \theta \coth \theta}{\operatorname{csch} \theta} \right) + (1 - \ln \operatorname{csch} \theta)(-\operatorname{csch} \theta \coth \theta) \\
 &= \operatorname{csch} \theta \coth \theta - (1 - \ln \operatorname{csch} \theta)(\operatorname{csch} \theta \coth \theta) = (\operatorname{csch} \theta \coth \theta)(1 - 1 + \ln \operatorname{csch} \theta) = (\operatorname{csch} \theta \coth \theta)(\ln \operatorname{csch} \theta)
 \end{aligned}$$

$$\begin{aligned}
 21. \quad y &= \ln \cosh v - \frac{1}{2} \tanh^2 v \Rightarrow \frac{dy}{dv} = \frac{\sinh v}{\cosh v} - \left(\frac{1}{2} \right) (2 \tanh v) (\operatorname{sech}^2 v) = \tanh v - (\tanh v) (\operatorname{sech}^2 v) \\
 &= (\tanh v) (1 - \operatorname{sech}^2 v) = (\tanh v) (\tanh^2 v) = \tanh^3 v
 \end{aligned}$$

$$\begin{aligned}
 22. \quad y &= \ln \sinh v - \frac{1}{2} \coth^2 v \Rightarrow \frac{dy}{dv} = \frac{\cosh v}{\sinh v} - \left(\frac{1}{2} \right) (2 \coth v) (-\operatorname{csch}^2 v) = \coth v + (\coth v) (\operatorname{csch}^2 v) \\
 &= (\coth v) (1 + \operatorname{csch}^2 v) = (\coth v) (\coth^2 v) = \coth^3 v
 \end{aligned}$$

$$23. \quad y = (x^2 + 1) \operatorname{sech}(\ln x) = (x^2 + 1) \left(\frac{2}{e^{\ln x} + e^{-\ln x}} \right) = (x^2 + 1) \left(\frac{2}{x + x^{-1}} \right) = (x^2 + 1) \left(\frac{2x}{x^2 + 1} \right) = 2x \Rightarrow \frac{dy}{dx} = 2$$

$$\begin{aligned}
 24. \quad y &= (4x^2 - 1) \operatorname{csch}(\ln 2x) = (4x^2 - 1) \left(\frac{2}{e^{\ln 2x} + e^{-\ln 2x}} \right) = (4x^2 - 1) \left(\frac{2}{2x + (2x)^{-1}} \right) = (4x^2 - 1) \left(\frac{4x}{4x^2 - 1} \right) \\
 &= 4x \Rightarrow \frac{dy}{dx} = 4
 \end{aligned}$$

$$25. \quad y = \sinh^{-1} \sqrt{x} = \sinh^{-1}(x^{1/2}) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{2} \right) x^{-1/2}}{\sqrt{1 + (x^{1/2})^2}} = \frac{1}{2\sqrt{x} \sqrt{1 + x}} = \frac{1}{2\sqrt{x(1 + x)}}$$

$$26. \quad y = \cosh^{-1} 2\sqrt{x+1} = \cosh^{-1} \left(2(x+1)^{1/2} \right) \Rightarrow \frac{dy}{dx} = \frac{(2) \left(\frac{1}{2} \right) (x+1)^{-1/2}}{\sqrt{\left[2(x+1)^{1/2} \right]^2 - 1}} = \frac{1}{\sqrt{x+1} \sqrt{4x+3}} = \frac{1}{\sqrt{4x^2 + 7x + 3}}$$

$$27. \quad y = (1 - \theta) \tanh^{-1} \theta \Rightarrow \frac{dy}{d\theta} = (1 - \theta) \left(\frac{1}{1 - \theta^2} \right) + (-1) \tanh^{-1} \theta = \frac{1}{1 + \theta} - \tanh^{-1} \theta$$

$$\begin{aligned}
 28. \quad y &= (\theta^2 + 2\theta) \tanh^{-1}(\theta + 1) \Rightarrow \frac{dy}{d\theta} = (\theta^2 + 2\theta) \left[\frac{1}{1 - (\theta + 1)^2} \right] + (2\theta + 2) \tanh^{-1}(\theta + 1) \\
 &= \frac{\theta^2 + 2\theta}{-\theta^2 - 2\theta} + (2\theta + 2) \tanh^{-1}(\theta + 1) = (2\theta + 2) \tanh^{-1}(\theta + 1) - 1
 \end{aligned}$$

$$29. \quad y = (1 - t) \coth^{-1} \sqrt{t} = (1 - t) \coth^{-1}(t^{1/2}) \Rightarrow \frac{dy}{dt} = (1 - t) \left[\frac{\left(\frac{1}{2} \right) t^{-1/2}}{1 - (t^{1/2})^2} \right] + (-1) \coth^{-1}(t^{1/2}) = \frac{1}{2\sqrt{t}} - \coth^{-1} \sqrt{t}$$

$$30. \quad y = (1 - t^2) \coth^{-1} t \Rightarrow \frac{dy}{dt} = (1 - t^2) \left(\frac{1}{1 - t^2} \right) + (-2t) \coth^{-1} t = 1 - 2t \coth^{-1} t$$

$$\begin{aligned}
 31. \quad y &= \cos^{-1} x - x \operatorname{sech}^{-1} x \Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1 - x^2}} - \left[x \left(\frac{-1}{x\sqrt{1 - x^2}} \right) + (1) \operatorname{sech}^{-1} x \right] = \frac{-1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - x^2}} - \operatorname{sech}^{-1} x \\
 &= -\operatorname{sech}^{-1} x
 \end{aligned}$$

$$32. y = \ln x + \sqrt{1-x^2} \operatorname{sech}^{-1} x = \ln x + (1-x^2)^{1/2} \operatorname{sech}^{-1} x \Rightarrow \frac{dy}{dx}$$

$$= \frac{1}{x} + (1-x^2)^{1/2} \left(\frac{-1}{x\sqrt{1-x^2}} \right) + \left(\frac{1}{2} \right) (1-x^2)^{-1/2} (-2x) \operatorname{sech}^{-1} x = \frac{1}{x} - \frac{1}{x} - \frac{x}{\sqrt{1-x^2}} \operatorname{sech}^{-1} x = \frac{-x}{\sqrt{1-x^2}} \operatorname{sech}^{-1} x$$

$$33. y = \operatorname{csch}^{-1} \left(\frac{1}{2} \right)^\theta \Rightarrow \frac{dy}{d\theta} = - \frac{\left[\ln \left(\frac{1}{2} \right) \right] \left(\frac{1}{2} \right)^\theta}{\left(\frac{1}{2} \right)^\theta \sqrt{1 + \left[\left(\frac{1}{2} \right)^\theta \right]^2}} = - \frac{\ln(1) - \ln(2)}{\sqrt{1 + \left(\frac{1}{2} \right)^{2\theta}}} = \frac{\ln 2}{\sqrt{1 + \left(\frac{1}{2} \right)^{2\theta}}}$$

$$34. y = \operatorname{csch}^{-1} 2^\theta \Rightarrow \frac{dy}{d\theta} = - \frac{(\ln 2) 2^\theta}{2^\theta \sqrt{1 + (2^\theta)^2}} = \frac{-\ln 2}{\sqrt{1 + 2^{2\theta}}}$$

$$35. y = \sinh^{-1}(\tan x) \Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{\sqrt{1 + (\tan x)^2}} = \frac{\sec^2 x}{\sqrt{\sec^2 x}} = \frac{\sec^2 x}{|\sec x|} = \frac{|\sec x| |\sec x|}{|\sec x|} = |\sec x|$$

$$36. y = \cosh^{-1}(\sec x) \Rightarrow \frac{dy}{dx} = \frac{(\sec x)(\tan x)}{\sqrt{\sec^2 x - 1}} = \frac{(\sec x)(\tan x)}{\sqrt{\tan^2 x}} = \frac{(\sec x)(\tan x)}{|\tan x|} = \sec x, 0 < x < \frac{\pi}{2}$$

$$37. (a) \text{ If } y = \tan^{-1}(\sinh x) + C, \text{ then } \frac{dy}{dx} = \frac{\cosh x}{1 + \sinh^2 x} = \frac{\cosh x}{\cosh^2 x} = \operatorname{sech} x, \text{ which verifies the formula}$$

$$(b) \text{ If } y = \sin^{-1}(\tanh x) + C, \text{ then } \frac{dy}{dx} = \frac{\operatorname{sech}^2 x}{\sqrt{1 - \tanh^2 x}} = \frac{\operatorname{sech}^2 x}{\operatorname{sech} x} = \operatorname{sech} x, \text{ which verifies the formula}$$

$$38. \text{ If } y = \frac{x^2}{2} \operatorname{sech}^{-1} x - \frac{1}{2} \sqrt{1-x^2} + C, \text{ then } \frac{dy}{dx} = x \operatorname{sech}^{-1} x + \frac{x^2}{2} \left(\frac{-1}{x\sqrt{1-x^2}} \right) + \frac{2x}{4\sqrt{1-x^2}} = x \operatorname{sech}^{-1} x,$$

which verifies the formula

$$39. \text{ If } y = \frac{x^2-1}{2} \coth^{-1} x + \frac{x}{2} + C, \text{ then } \frac{dy}{dx} = x \coth^{-1} x + \left(\frac{x^2-1}{2} \right) \left(\frac{1}{1-x^2} \right) + \frac{1}{2} = x \coth^{-1} x, \text{ which verifies the formula}$$

$$40. \text{ If } y = x \tanh^{-1} x + \frac{1}{2} \ln(1-x^2) + C, \text{ then } \frac{dy}{dx} = \tanh^{-1} x + x \left(\frac{1}{1-x^2} \right) + \frac{1}{2} \left(\frac{-2x}{1-x^2} \right) = \tanh^{-1} x, \text{ which verifies the formula}$$

$$41. \int \sinh 2x \, dx = \frac{1}{2} \int \sinh u \, du, \text{ where } u = 2x \text{ and } du = 2 \, dx$$

$$= \frac{\cosh u}{2} + C = \frac{\cosh 2x}{2} + C$$

$$42. \int \sinh \frac{x}{5} \, dx = 5 \int \sinh u \, du, \text{ where } u = \frac{x}{5} \text{ and } du = \frac{1}{5} \, dx$$

$$= 5 \cosh u + C = 5 \cosh \frac{x}{5} + C$$

43. $\int 6 \cosh\left(\frac{x}{2} - \ln 3\right) dx = 12 \int \cosh u \, du$, where $u = \frac{x}{2} - \ln 3$ and $du = \frac{1}{2} dx$
 $= 12 \sinh u + C = 12 \sinh\left(\frac{x}{2} - \ln 3\right) + C$
44. $\int 4 \cosh(3x - \ln 2) dx = \frac{4}{3} \int \cosh u \, du$, where $u = 3x - \ln 2$ and $du = 3 dx$
 $= \frac{4}{3} \sinh u + C = \frac{4}{3} \sinh(3x - \ln 2) + C$
45. $\int \tanh \frac{x}{7} dx = 7 \int \frac{\sinh u}{\cosh u} du$, where $u = \frac{x}{7}$ and $du = \frac{1}{7} dx$
 $= 7 \ln |\cosh u| + C_1 = 7 \ln \left| \cosh \frac{x}{7} \right| + C_1 = 7 \ln \left| \frac{e^{x/7} + e^{-x/7}}{2} \right| + C_1 = 7 \ln |e^{x/7} + e^{-x/7}| - 7 \ln 2 + C_1$
 $= 7 \ln (e^{x/7} + e^{-x/7}) + C$, since $e^{x/7} + e^{-x/7} > 0$ for all x .
46. $\int \coth \frac{\theta}{\sqrt{3}} d\theta = \sqrt{3} \int \frac{\cosh u}{\sinh u} du$, where $u = \frac{\theta}{\sqrt{3}}$ and $du = \frac{d\theta}{\sqrt{3}}$
 $= \sqrt{3} \ln |\sinh u| + C_1 = \sqrt{3} \ln \left| \sinh \frac{\theta}{\sqrt{3}} \right| + C_1 = \sqrt{3} \ln \left| \frac{e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}}{2} \right| + C_1$
 $= \sqrt{3} \ln |e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}| - \sqrt{3} \ln 2 + C_1 = \sqrt{3} \ln |e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}| + C$
47. $\int \operatorname{sech}^2\left(x - \frac{1}{2}\right) dx = \int \operatorname{sech}^2 u \, du$, where $u = \left(x - \frac{1}{2}\right)$ and $du = dx$
 $= \tanh u + C = \tanh\left(x - \frac{1}{2}\right) + C$
48. $\int \operatorname{csch}^2(5 - x) dx = - \int \operatorname{csch}^2 u \, du$, where $u = (5 - x)$ and $du = -dx$
 $= -(-\coth u) + C = \coth u + C = \coth(5 - x) + C$
49. $\int \frac{\operatorname{sech} \sqrt{t} \tanh \sqrt{t}}{\sqrt{t}} dt = 2 \int \operatorname{sech} u \tanh u \, du$, where $u = \sqrt{t} = t^{1/2}$ and $du = \frac{dt}{2\sqrt{t}}$
 $= 2(-\operatorname{sech} u) + C = -2 \operatorname{sech} \sqrt{t} + C$
50. $\int \frac{\operatorname{csch}(\ln t) \coth(\ln t)}{t} dt = \int \operatorname{csch} u \coth u \, du$, where $u = \ln t$ and $du = \frac{dt}{t}$
 $= -\operatorname{csch} u + C = -\operatorname{csch}(\ln t) + C$
51. $\int_{\ln 2}^{\ln 4} \coth x \, dx = \int_{\ln 2}^{\ln 4} \frac{\cosh x}{\sinh x} dx = \int_{3/4}^{15/8} \frac{1}{u} du = [\ln |u|]_{3/4}^{15/8} = \ln \left| \frac{15}{8} \right| - \ln \left| \frac{3}{4} \right| = \ln \left| \frac{15}{8} \cdot \frac{4}{3} \right| = \ln \frac{5}{2}$,
 where $u = \sinh x$, $du = \cosh x \, dx$, the lower limit is $\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - \left(\frac{1}{2}\right)}{2} = \frac{3}{4}$ and the upper

$$\text{limit is } \sinh(\ln 4) = \frac{e^{\ln 4} - e^{-\ln 4}}{2} = \frac{4 - \left(\frac{1}{4}\right)}{2} = \frac{15}{8}$$

$$52. \int_0^{\ln 2} \tanh 2x \, dx = \int_0^{\ln 2} \frac{\sinh 2x}{\cosh 2x} \, dx = \frac{1}{2} \int_1^{17/8} \frac{1}{u} \, du = \frac{1}{2} [\ln |u|]_1^{17/8} = \frac{1}{2} \left[\ln \left(\frac{17}{8} \right) - \ln 1 \right] = \frac{1}{2} \ln \frac{17}{8}, \text{ where}$$

$u = \cosh 2x$, $du = 2 \sinh(2x) \, dx$, the lower limit is $\cosh 0 = 1$ and the upper limit is $\cosh(2 \ln 2) = \cosh(\ln 4)$

$$= \frac{e^{\ln 4} + e^{-\ln 4}}{2} = \frac{4 + \left(\frac{1}{4}\right)}{2} = \frac{17}{8}$$

$$53. \int_{-\ln 4}^{-\ln 2} 2e^\theta \cosh \theta \, d\theta = \int_{-\ln 4}^{-\ln 2} 2e^\theta \left(\frac{e^\theta + e^{-\theta}}{2} \right) d\theta = \int_{-\ln 4}^{-\ln 2} (e^{2\theta} + 1) \, d\theta = \left[\frac{e^{2\theta}}{2} + \theta \right]_{-\ln 4}^{-\ln 2}$$

$$= \left(\frac{e^{-2 \ln 2}}{2} - \ln 2 \right) - \left(\frac{e^{-2 \ln 4}}{2} - \ln 4 \right) = \left(\frac{1}{8} - \ln 2 \right) - \left(\frac{1}{32} - \ln 4 \right) = \frac{3}{32} - \ln 2 + 2 \ln 2 = \frac{3}{32} + \ln 2$$

$$54. \int_0^{\ln 2} 4e^{-\theta} \sinh \theta \, d\theta = \int_0^{\ln 2} 4e^{-\theta} \left(\frac{e^\theta - e^{-\theta}}{2} \right) d\theta = 2 \int_0^{\ln 2} (1 - e^{-2\theta}) \, d\theta = 2 \left[\theta + \frac{e^{-2\theta}}{2} \right]_0^{\ln 2}$$

$$= 2 \left[\left(\ln 2 + \frac{e^{-2 \ln 2}}{2} \right) - \left(0 + \frac{e^{-0}}{2} \right) \right] = 2 \left(\ln 2 + \frac{1}{8} - \frac{1}{2} \right) = 2 \ln 2 + \frac{1}{4} - 1 = \ln 4 - \frac{3}{4}$$

$$55. \int_{-\pi/4}^{\pi/4} \cosh(\tan \theta) \sec^2 \theta \, d\theta = \int_{-1}^1 \cosh u \, du = [\sinh u]_{-1}^1 = \sinh(1) - \sinh(-1) = \left(\frac{e^1 - e^{-1}}{2} \right) - \left(\frac{e^{-1} - e^1}{2} \right)$$

$$= \frac{e - e^{-1} - e^{-1} + e}{2} = e - e^{-1}, \text{ where } u = \tan \theta, \, du = \sec^2 \theta \, d\theta, \text{ the lower limit is } \tan\left(-\frac{\pi}{4}\right) = -1 \text{ and the upper}$$

limit is $\tan\left(\frac{\pi}{4}\right) = 1$

$$56. \int_0^{\pi/2} 2 \sinh(\sin \theta) \cos \theta \, d\theta = 2 \int_0^1 \sinh u \, du = 2 [\cosh u]_0^1 = 2(\cosh 1 - \cosh 0) = 2 \left(\frac{e + e^{-1}}{2} - 1 \right)$$

$$= e + e^{-1} - 2, \text{ where } u = \sin \theta, \, du = \cos \theta \, d\theta, \text{ the lower limit is } \sin 0 = 0 \text{ and the upper limit is } \sin\left(\frac{\pi}{2}\right) = 1$$

$$57. \int_1^2 \frac{\cosh(\ln t)}{t} \, dt = \int_0^{\ln 2} \cosh u \, du = [\sinh u]_0^{\ln 2} = \sinh(\ln 2) - \sinh(0) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} - 0 = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}, \text{ where}$$

$u = \ln t$, $du = \frac{1}{t} \, dt$, the lower limit is $\ln 1 = 0$ and the upper limit is $\ln 2$

$$58. \int_1^4 \frac{8 \cosh \sqrt{x}}{\sqrt{x}} \, dx = 16 \int_1^2 \cosh u \, du = 16 [\sinh u]_1^2 = 16(\sinh 2 - \sinh 1) = 16 \left[\left(\frac{e^2 - e^{-2}}{2} \right) - \left(\frac{e - e^{-1}}{2} \right) \right]$$

$$= 8(e^2 - e^{-2} - e + e^{-1}), \text{ where } u = \sqrt{x} = x^{1/2}, \, du = \frac{1}{2} x^{-1/2} = \frac{dx}{2\sqrt{x}}, \text{ the lower limit is } \sqrt{1} = 1 \text{ and the upper}$$

limit is $\sqrt{4} = 2$

$$\begin{aligned}
 59. \quad \int_{-\ln 2}^0 \cosh^2\left(\frac{x}{2}\right) dx &= \int_{-\ln 2}^0 \frac{\cosh x + 1}{2} dx = \frac{1}{2} \int_{-\ln 2}^0 (\cosh x + 1) dx = \frac{1}{2} [\sinh x + x]_{-\ln 2}^0 \\
 &= \frac{1}{2} [(\sinh 0 + 0) - (\sinh(-\ln 2) - \ln 2)] = \frac{1}{2} \left[(0 + 0) - \left(\frac{e^{-\ln 2} - e^{\ln 2}}{2} - \ln 2 \right) \right] = \frac{1}{2} \left[-\frac{\left(\frac{1}{2}\right) - 2}{2} + \ln 2 \right] \\
 &= \frac{1}{2} \left(1 - \frac{1}{4} + \ln 2 \right) = \frac{3}{8} + \frac{1}{2} \ln 2 = \frac{3}{8} + \ln \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 60. \quad \int_0^{\ln 10} 4 \sinh^2\left(\frac{x}{2}\right) dx &= \int_0^{\ln 10} 4 \left(\frac{\cosh x - 1}{2} \right) dx = 2 \int_0^{\ln 10} (\cosh x - 1) dx = 2 [\sinh x - x]_0^{\ln 10} \\
 &= 2[(\sinh(\ln 10) - \ln 10) - (\sinh 0 - 0)] = e^{\ln 10} - e^{-\ln 10} - 2 \ln 10 = 10 - \frac{1}{10} - 2 \ln 10 = 9.9 - 2 \ln 10
 \end{aligned}$$

$$61. \sinh^{-1}\left(\frac{-5}{12}\right) = \ln\left(-\frac{5}{12} + \sqrt{\frac{25}{144} + 1}\right) = \ln\left(\frac{2}{3}\right) \qquad 62. \cosh^{-1}\left(\frac{5}{3}\right) = \ln\left(\frac{5}{3} + \sqrt{\frac{25}{9} - 1}\right) = \ln 3$$

$$63. \tanh^{-1}\left(-\frac{1}{2}\right) = \frac{1}{2} \ln\left(\frac{1 - (1/2)}{1 + (1/2)}\right) = -\frac{\ln 3}{2} \qquad 64. \coth^{-1}\left(\frac{5}{4}\right) = \frac{1}{2} \ln\left(\frac{(9/4)}{(1/4)}\right) = \frac{1}{2} \ln 9 = \ln 3$$

$$65. \operatorname{sech}^{-1}\left(\frac{3}{5}\right) = \ln\left(\frac{1 + \sqrt{1 - (9/25)}}{(3/5)}\right) = \ln 3 \qquad 66. \operatorname{csch}^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \ln\left(-\sqrt{3} + \frac{\sqrt{4/3}}{(1/\sqrt{3})}\right) = \ln(-\sqrt{3} + 2)$$

$$67. (a) \int_0^{2\sqrt{3}} \frac{dx}{\sqrt{4+x^2}} = \left[\sinh^{-1} \frac{x}{2} \right]_0^{2\sqrt{3}} = \sinh^{-1} \sqrt{3} - \sinh^{-1} 0 = \sinh^{-1} \sqrt{3}$$

$$(b) \sinh^{-1} \sqrt{3} = \ln(\sqrt{3} + \sqrt{3+1}) = \ln(\sqrt{3} + 2)$$

$$68. (a) \int_0^{1/3} \frac{6 dx}{\sqrt{1+9x^2}} = 2 \int_0^1 \frac{dx}{\sqrt{a^2+u^2}}, \text{ where } u = 3x, du = 3 dx, a = 1$$

$$= [2 \sinh^{-1} u]_0^1 = 2(\sinh^{-1} 1 - \sinh^{-1} 0) = 2 \sinh^{-1} 1$$

$$(b) 2 \sinh^{-1} 1 = 2 \ln(1 + \sqrt{1^2 + 1}) = 2 \ln(1 + \sqrt{2})$$

$$69. (a) \int_{5/4}^2 \frac{1}{1-x^2} dx = [\coth^{-1} x]_{5/4}^2 = \coth^{-1} 2 - \coth^{-1} \frac{5}{4}$$

$$(b) \coth^{-1} 2 - \coth^{-1} \frac{5}{4} = \frac{1}{2} \left[\ln 3 - \ln\left(\frac{9/4}{1/4}\right) \right] = \frac{1}{2} \ln \frac{1}{3}$$

$$70. (a) \int_0^{1/2} \frac{1}{1-x^2} dx = [\tanh^{-1} x]_0^{1/2} = \tanh^{-1} \frac{1}{2} - \tanh^{-1} 0 = \tanh^{-1} \frac{1}{2}$$

$$(b) \tanh^{-1} \frac{1}{2} = \frac{1}{2} \ln \left(\frac{1 + (1/2)}{1 - (1/2)} \right) = \frac{1}{2} \ln 3$$

$$71. (a) \int_{1/5}^{3/13} \frac{dx}{x\sqrt{1-16x^2}} = \int_{4/5}^{12/13} \frac{du}{u\sqrt{a^2-u^2}}, \text{ where } u = 4x, du = 4 dx, a = 1$$

$$= [-\operatorname{sech}^{-1} u]_{4/5}^{12/13} = -\operatorname{sech}^{-1} \frac{12}{13} + \operatorname{sech}^{-1} \frac{4}{5}$$

$$(b) -\operatorname{sech}^{-1} \frac{12}{13} + \operatorname{sech}^{-1} \frac{4}{5} = -\ln \left(\frac{1 + \sqrt{1 - (12/13)^2}}{(12/13)} \right) + \ln \left(\frac{1 + \sqrt{1 - (4/5)^2}}{(4/5)} \right)$$

$$= -\ln \left(\frac{13 + \sqrt{169 - 144}}{12} \right) + \ln \left(\frac{5 + \sqrt{25 - 16}}{4} \right) = \ln \left(\frac{5+3}{4} \right) - \ln \left(\frac{13+5}{12} \right) = \ln 2 - \ln \frac{3}{2}$$

$$= \ln \left(2 \cdot \frac{2}{3} \right) = \ln \frac{4}{3}$$

$$72. (a) \int_1^2 \frac{dx}{x\sqrt{4+x^2}} = \left[-\frac{1}{2} \operatorname{csch}^{-1} \left| \frac{x}{2} \right| \right]_1^2 = -\frac{1}{2} \left(\operatorname{csch}^{-1} 1 - \operatorname{csch}^{-1} \frac{1}{2} \right) = \frac{1}{2} \left(\operatorname{csch}^{-1} \frac{1}{2} - \operatorname{csch}^{-1} 1 \right)$$

$$(b) \frac{1}{2} \left(\operatorname{csch}^{-1} \frac{1}{2} - \operatorname{csch}^{-1} 1 \right) = \frac{1}{2} \left[\ln \left(2 + \frac{\sqrt{5/4}}{(1/2)} \right) - \ln(1 + \sqrt{2}) \right] = \frac{1}{2} \ln \left(\frac{2 + \sqrt{5}}{1 + \sqrt{2}} \right)$$

$$73. (a) \int_0^{\pi} \frac{\cos x}{\sqrt{1 + \sin^2 x}} dx = \int_0^0 \frac{1}{\sqrt{1 + u^2}} du = [\sinh^{-1} u]_0^0 = \sinh^{-1} 0 - \sinh^{-1} 0 = 0, \text{ where } u = \sin x, du = \cos x dx$$

$$(b) \sinh^{-1} 0 - \sinh^{-1} 0 = \ln(0 + \sqrt{0+1}) - \ln(0 + \sqrt{0+1}) = 0$$

$$74. (a) \int_1^e \frac{dx}{x\sqrt{1 + (\ln x)^2}} = \int_0^1 \frac{du}{\sqrt{a^2 + u^2}}, \text{ where } u = \ln x, du = \frac{1}{x} dx, a = 1$$

$$= [\sinh^{-1} u]_0^1 = \sinh^{-1} 1 - \sinh^{-1} 0 = \sinh^{-1} 1$$

$$(b) \sinh^{-1} 1 - \sinh^{-1} 0 = \ln(1 + \sqrt{1^2 + 1}) - \ln(0 + \sqrt{0^2 + 1}) = \ln(1 + \sqrt{2})$$

$$75. (a) \text{ Let } E(x) = \frac{f(x) + f(-x)}{2} \text{ and } O(x) = \frac{f(x) - f(-x)}{2}. \text{ Then } E(x) + O(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

$$= \frac{2f(x)}{2} = f(x). \text{ Also, } E(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(x) + f(-x)}{2} = E(x) \Rightarrow E(x) \text{ is even, and}$$

$$O(-x) = \frac{f(-x) - f(-(-x))}{2} = -\frac{f(x) - f(-x)}{2} = -O(x) \Rightarrow O(x) \text{ is odd. Consequently, } f(x) \text{ can be written as a sum of an even and an odd function.}$$

$$(b) f(x) = \frac{f(x) + f(-x)}{2} \text{ because } \frac{f(x) - f(-x)}{2} = 0 \text{ and } f(x) = \frac{f(x) - f(-x)}{2} \text{ because } \frac{f(x) + f(-x)}{2} = 0; \text{ thus}$$

$$f(x) = \frac{2f(x)}{2} + 0 \text{ and } f(x) = 0 + \frac{2f(x)}{2}$$

$$76. y = \sinh^{-1} x \Rightarrow x = \sinh y \Rightarrow x = \frac{e^y - e^{-y}}{2} \Rightarrow 2x = e^y - \frac{1}{e^y} \Rightarrow 2xe^y = e^{2y} - 1 \Rightarrow e^{2y} - 2xe^y - 1 = 0$$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} \Rightarrow e^y = x + \sqrt{x^2 + 1} \Rightarrow \sinh^{-1} x = y = \ln(x + \sqrt{x^2 + 1}). \text{ Since } e^y > 0, \text{ we cannot}$$

choose $e^y = x - \sqrt{x^2 + 1}$ because $x - \sqrt{x^2 + 1} < 0$.

$$77. (a) m \frac{dv}{dt} = mg - kv^2 \Rightarrow \frac{m \frac{dv}{dt}}{mg - kv^2} = 1 \Rightarrow \frac{\frac{1}{g} \frac{dv}{dt}}{1 - \frac{kv^2}{mg}} = 1 \Rightarrow \frac{\sqrt{\frac{k}{mg}} dv}{1 - \sqrt{v \left(\frac{k}{mg} \right)^2}} = \sqrt{\frac{kg}{m}} dt \Rightarrow \int \frac{\sqrt{\frac{k}{mg}} dv}{1 - \sqrt{v \left(\frac{k}{mg} \right)^2}} dv$$

$$= \int \sqrt{\frac{kg}{m}} dt \Rightarrow \tanh^{-1} \left(\sqrt{\frac{k}{mg}} v \right) = \sqrt{\frac{kg}{m}} t + C \Rightarrow v = \sqrt{\frac{mg}{k}} \tanh \left(\sqrt{\frac{kg}{m}} t + C \right); v(0) = 0 \Rightarrow C = 0$$

$$\Rightarrow v = \sqrt{\frac{mg}{k}} \tanh \left(\sqrt{\frac{kg}{m}} t \right)$$

$$(b) \lim_{t \rightarrow \infty} v = \lim_{t \rightarrow \infty} \sqrt{\frac{mg}{k}} \tanh \left(\sqrt{\frac{kg}{m}} t \right) = \sqrt{\frac{mg}{k}} \lim_{t \rightarrow \infty} \tanh \left(\sqrt{\frac{kg}{m}} t \right) = \sqrt{\frac{mg}{k}} (1) = \sqrt{\frac{mg}{k}}$$

$$(c) \sqrt{\frac{160}{0.005}} = \sqrt{\frac{1,600,000}{5}} = \frac{400}{\sqrt{5}} = 80\sqrt{5} \approx 178.89 \text{ ft/sec}$$

$$78. (a) s(t) = a \cos kt + b \sin kt \Rightarrow \frac{ds}{dt} = -ak \sin kt + bk \cos kt \Rightarrow \frac{d^2s}{dt^2} = -ak^2 \cos kt - bk^2 \sin kt$$

$= -k^2(a \cos kt + b \sin kt) = -k^2 s(t) \Rightarrow$ acceleration is proportional to s . The negative constant $-k^2$ implies that the acceleration is directed toward the origin.

$$(b) s(t) = a \cosh kt + b \sinh kt \Rightarrow \frac{ds}{dt} = ak \sinh kt + bk \cosh kt \Rightarrow \frac{d^2s}{dt^2} = ak^2 \cosh kt + bk^2 \sinh kt$$

$= k^2(a \cosh kt + b \sinh kt) = k^2 s(t) \Rightarrow$ acceleration is proportional to s . The positive constant k^2 implies that the acceleration is directed away from the origin.

$$79. \frac{dy}{dx} = \frac{-1}{x\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} \Rightarrow y = \int \frac{-1}{x\sqrt{1-x^2}} dx + \int \frac{x}{\sqrt{1-x^2}} dx \Rightarrow y = \operatorname{sech}^{-1}(x) - \sqrt{1-x^2} + C; x = 1 \text{ and}$$

$$y = 0 \Rightarrow C = 0 \Rightarrow y = \operatorname{sech}^{-1}(x) - \sqrt{1-x^2}$$

$$80. \text{ To find the length of the curve: } y = \frac{1}{a} \cosh ax \Rightarrow y' = \sinh ax \Rightarrow s = \int_0^b \sqrt{1 + (\sinh ax)^2} dx$$

$$\Rightarrow s = \int_0^b \cosh ax dx = \left[\frac{1}{a} \sinh ax \right]_0^b = \frac{1}{a} \sinh ab. \text{ Then the area under the curve is } A = \int_0^b \frac{1}{a} \cosh ax dx$$

$= \left[\frac{1}{a^2} \sinh ax \right]_0^b = \frac{1}{a^2} \sinh ab = \left(\frac{1}{a} \right) \left(\frac{1}{a} \sinh ab \right)$ which is the area of the rectangle of height $\frac{1}{a}$ and length s as claimed.

$$81. V = \pi \int_0^2 (\cosh^2 x - \sinh^2 x) dx = \pi \int_0^2 1 dx = 2\pi$$

$$82. V = 2\pi \int_0^{\ln \sqrt{3}} \operatorname{sech}^2 x dx = 2\pi [\tanh x]_0^{\ln \sqrt{3}} = 2\pi \left[\frac{\sqrt{3} - (1/\sqrt{3})}{\sqrt{3} + (1/\sqrt{3})} \right] = \pi$$

$$83. y = \frac{1}{2} \cosh 2x \Rightarrow y' = \sinh 2x \Rightarrow L = \int_0^{\ln \sqrt{5}} \sqrt{1 + (\sinh 2x)^2} dx = \int_0^{\ln \sqrt{5}} \cosh 2x dx = \left[\frac{1}{2} \sinh 2x \right]_0^{\ln \sqrt{5}} \\ = \left[\frac{1}{2} \left(\frac{e^{2x} - e^{-2x}}{2} \right) \right]_0^{\ln \sqrt{5}} = \frac{1}{4} \left(5 - \frac{1}{5} \right) = \frac{6}{5}$$

84. (a) Let the point located at $(\cosh x, 0)$ be called T. Then $A(u)$ = area of the triangle $\triangle OTP$ minus the area

$$\text{under the curve } y = \sqrt{x^2 - 1} \text{ from A to T} \Rightarrow A(u) = \frac{1}{2} \cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2 - 1} dx.$$

$$(b) A(u) = \frac{1}{2} \cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2 - 1} dx \Rightarrow A'(u) = \frac{1}{2} (\cosh^2 u + \sinh^2 u) - (\sqrt{\cosh^2 u - 1}) (\sinh u) \\ = \frac{1}{2} \cosh^2 u + \frac{1}{2} \sinh^2 u - \sinh^2 u = \frac{1}{2} (\cosh^2 u - \sinh^2 u) = \left(\frac{1}{2} \right) (1) = \frac{1}{2}$$

$$(c) A'(u) = \frac{1}{2} \Rightarrow A(u) = \frac{u}{2} + C, \text{ and from part (a) we have } A(0) = 0 \Rightarrow C = 0 \Rightarrow A(u) = \frac{u}{2} \Rightarrow u = 2A$$

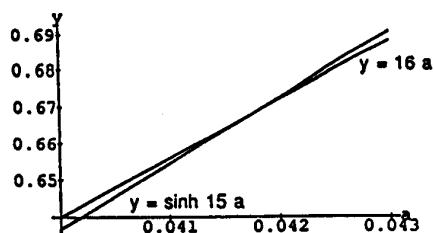
$$85. (a) y = \frac{H}{W} \cosh \left(\frac{W}{H} x \right) \Rightarrow \tan \phi = \frac{dy}{dx} = \left(\frac{H}{W} \right) \left[\frac{W}{H} \sinh \left(\frac{W}{H} x \right) \right] = \sinh \left(\frac{W}{H} x \right)$$

$$(b) \text{ The tension at P is given by } T \cos \phi = H \Rightarrow T = H \sec \phi = H \sqrt{1 + \tan^2 \phi} = H \sqrt{1 + \left(\sinh \frac{W}{H} x \right)^2} \\ = H \cosh \left(\frac{W}{H} x \right) = w \left(\frac{H}{W} \right) \cosh \left(\frac{W}{H} x \right) = wy$$

$$86. s = \frac{1}{a} \sinh ax \Rightarrow \sinh ax = as \Rightarrow ax = \sinh^{-1} as \Rightarrow x = \frac{1}{a} \sinh^{-1} as; y = \frac{1}{a} \cosh ax = \frac{1}{a} \sqrt{\cosh^2 ax} \\ = \frac{1}{a} \sqrt{\sinh^2 ax + 1} = \frac{1}{a} \sqrt{a^2 s^2 + 1} = \sqrt{s^2 + \frac{1}{a^2}}$$

$$87. (a) \text{ Since the cable is 32 ft long, } s = 16 \text{ and } x = 15. \text{ From Exercise 88, } x = \frac{1}{a} \sinh^{-1} as \Rightarrow 15a = \sinh^{-1} 16a \\ \Rightarrow \sinh 15a = 16a.$$

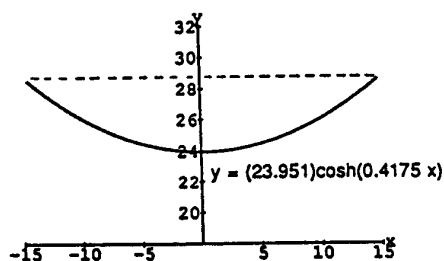
- (b) The intersection is near $(0.042, 0.672)$.



- (c) Newton's method indicates that at $a \approx 0.0417525$ the curves $y = 16a$ and $y = \sinh 15a$ intersect.

(d) $T = wy \approx (2 \text{ lb})\left(\frac{1}{0.0417525}\right) \approx 47.90 \text{ lb}$

- (e) The sag is about 4.8 ft.



CHAPTER 6 PRACTICE EXERCISES

- $$\int e^x \sin(e^x) dx = \int \sin u du, \text{ where } u = e^x \text{ and } du = e^x dx,$$

$$= -\cos u + C = -\cos(e^x) + C$$
- $$\int e^t \cos(3e^t - 2) dt = \frac{1}{3} \int \cos u du, \text{ where } u = 3e^t - 2 \text{ and } du = 3e^t dt,$$

$$= \frac{1}{3} \sin u + C = \frac{1}{3} \sin(3e^t - 2) + C$$
- $$\int e^x \sec^2(e^x - 7) dx = \int \sec^2 u du, \text{ where } u = e^x - 7 \text{ and } du = e^x dx,$$

$$= \tan u + C = \tan(e^x - 7) + C$$
- $$\int e^y \csc(e^y + 1) \cot(e^y + 1) dy = \int \csc u \cot u du, \text{ where } u = e^y + 1 \text{ and } du = e^y dy,$$

$$= -\csc u + C = -\csc(e^y + 1) + C$$
- $$\int (\sec^2 x) e^{\tan x} dx = \int e^u du, \text{ where } u = \tan x \text{ and } du = \sec^2 x dx,$$

$$= e^u + C = e^{\tan x} + C$$

$$\begin{aligned} 6. \quad \int (\csc^2 x) e^{\cot x} dx &= - \int e^u du, \text{ where } u = \cot x \text{ and } du = -\csc^2 x dx, \\ &= -e^u + C = -e^{\cot x} + C \end{aligned}$$

$$\begin{aligned} 7. \quad \int_{-1}^1 \frac{1}{3x-4} dx &= \frac{1}{3} \int_{-7}^{-1} \frac{1}{u} du, \text{ where } u = 3x-4, du = 3 dx; x = -1 \Rightarrow u = -7, x = 1 \Rightarrow u = -1, \\ &= \frac{1}{3} [\ln |u|]_{-7}^{-1} = \frac{1}{3} [\ln |-1| - \ln |-7|] = \frac{1}{3} [0 - \ln 7] = -\frac{\ln 7}{3} \end{aligned}$$

$$\begin{aligned} 8. \quad \int_1^e \frac{\sqrt{\ln x}}{x} dx &= \int_0^1 u^{1/2} du, \text{ where } u = \ln x, du = \frac{1}{x} dx; x = 1 \Rightarrow u = 0, x = e \Rightarrow u = 1, \\ &= \left[\frac{2}{3} u^{3/2} \right]_0^1 = \left[\frac{2}{3} 1^{3/2} - \frac{2}{3} 0^{3/2} \right] = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} 9. \quad \int_0^\pi \tan\left(\frac{x}{3}\right) dx &= \int_0^\pi \frac{\sin\left(\frac{x}{3}\right)}{\cos\left(\frac{x}{3}\right)} dx = -3 \int_1^{1/2} \frac{1}{u} du, \\ &\text{where } u = \cos\left(\frac{x}{3}\right), du = -\frac{1}{3} \sin\left(\frac{x}{3}\right) dx; x = 0 \Rightarrow u = 1, x = \pi \Rightarrow u = \frac{1}{2}, \\ &= -3 [\ln |u|]_1^{1/2} = -3 \left[\ln \left| \frac{1}{2} \right| - \ln |1| \right] = -3 \ln \frac{1}{2} = \ln 2^3 = \ln 8 \end{aligned}$$

$$\begin{aligned} 10. \quad \int_{1/6}^{1/4} 2 \cot \pi x dx &= 2 \int_{1/6}^{1/4} \frac{\cos \pi x}{\sin \pi x} dx = \frac{2}{\pi} \int_{1/2}^{1/\sqrt{2}} \frac{1}{u} du, \\ &\text{where } u = \sin \pi x, du = \pi \cos \pi x dx; x = \frac{1}{6} \Rightarrow u = \frac{1}{2}, x = \frac{1}{4} \Rightarrow u = \frac{1}{\sqrt{2}}, \\ &= \frac{2}{\pi} [\ln |u|]_{1/2}^{1/\sqrt{2}} = \frac{2}{\pi} \left[\ln \left| \frac{1}{\sqrt{2}} \right| - \ln \left| \frac{1}{2} \right| \right] = \frac{2}{\pi} \left[\ln 1 - \frac{1}{2} \ln 2 - \ln 1 + \ln 2 \right] = \frac{2}{\pi} \left[\frac{1}{2} \ln 2 \right] = \frac{\ln 2}{\pi} \end{aligned}$$

$$\begin{aligned} 11. \quad \int_0^4 \frac{2t}{t^2-25} dt &= \int_{-25}^{-9} \frac{1}{u} du, \text{ where } u = t^2-25, du = 2t dt; t = 0 \Rightarrow u = -25, t = 4 \Rightarrow u = -9, \\ &= [\ln |u|]_{-25}^{-9} = \ln |-9| - \ln |-25| = \ln 9 - \ln 25 = \ln \frac{9}{25} \end{aligned}$$

$$\begin{aligned} 12. \quad \int_{-\pi/2}^{\pi/6} \frac{\cos t}{1-\sin t} dt &= - \int_2^{1/2} \frac{1}{u} du, \text{ where } u = 1-\sin t, du = -\cos t dt; t = -\frac{\pi}{2} \Rightarrow u = 2, t = \frac{\pi}{6} \Rightarrow u = \frac{1}{2}, \\ &= -[\ln |u|]_2^{1/2} = -\left[\ln \left| \frac{1}{2} \right| - \ln |2| \right] = -\ln 1 + \ln 2 + \ln 2 = 2 \ln 2 = \ln 4 \end{aligned}$$

$$13. \int \frac{\tan(\ln v)}{v} dv = \int \tan u \, du = \int \frac{\sin u}{\cos u} du, \text{ where } u = \ln v \text{ and } du = \frac{1}{v} dv,$$

$$= -\ln |\cos u| + C = -\ln |\cos(\ln v)| + C$$

$$14. \int \frac{1}{v \ln v} dv = \int \frac{1}{u} du, \text{ where } u = \ln v \text{ and } du = \frac{1}{v} dv,$$

$$= \ln |u| + C = \ln |\ln v| + C$$

$$15. \int \frac{(\ln x)^{-3}}{x} dx = \int u^{-3} du, \text{ where } u = \ln x \text{ and } du = \frac{1}{x} dx,$$

$$= \frac{u^{-2}}{-2} + C = -\frac{1}{2}(\ln x)^{-2} + C$$

$$16. \int \frac{\ln(x-5)}{x-5} dx = \int u \, du, \text{ where } u = \ln(x-5) \text{ and } du = \frac{1}{x-5} dx,$$

$$= \frac{u^2}{2} + C = \frac{[\ln(x-5)]^2}{2} + C$$

$$17. \int \frac{1}{r} \csc^2(1 + \ln r) \, dr = \int \csc^2 u \, du, \text{ where } u = 1 + \ln r \text{ and } du = \frac{1}{r} dr,$$

$$= -\cot u + C = -\cot(1 + \ln r) + C$$

$$18. \int \frac{\cos(1 - \ln v)}{v} dv = - \int \cos u \, du, \text{ where } u = 1 - \ln v \text{ and } du = -\frac{1}{v} dv,$$

$$= -\sin u + C = -\sin(1 - \ln v) + C$$

$$19. \int x 3^{x^2} dx = \frac{1}{2} \int 3^u du, \text{ where } u = x^2 \text{ and } du = 2x \, dx,$$

$$= \frac{1}{2 \ln 3} (3^u) + C = \frac{1}{2 \ln 3} (3^{x^2}) + C$$

$$20. \int 2^{\tan x} \sec^2 x \, dx = \int 2^u du, \text{ where } u = \tan x \text{ and } du = \sec^2 x \, dx,$$

$$= \frac{1}{\ln 2} (2^u) + C = \frac{2^{\tan x}}{\ln 2} + C$$

$$21. \int_1^7 \frac{3}{x} dx = 3 \int_1^7 \frac{1}{x} dx = 3 [\ln |x|]_1^7 = 3(\ln 7 - \ln 1) = 3 \ln 7$$

$$22. \int_1^{32} \frac{1}{5x} dx = \frac{1}{5} \int_1^{32} \frac{1}{x} dx = \frac{1}{5} [\ln |x|]_1^{32} = \frac{1}{5} (\ln 32 - \ln 1) = \frac{1}{5} \ln 32 = \ln(\sqrt[5]{32}) = \ln 2$$

$$23. \int_{-2}^{-1} e^{-(x+1)} dx = - \int_1^0 e^u du, \text{ where } u = -(x+1), du = -dx; x = -2 \Rightarrow u = 1, x = -1 \Rightarrow u = 0$$

$$= -[e^u]_1^0 = -(e^0 - e^1) = e - 1$$

$$24. \int_{-\ln 2}^0 e^{2w} dw = \frac{1}{2} \int_{\ln(1/4)}^0 e^u du, \text{ where } u = 2w, du = 2 dw; w = -\ln 2 \Rightarrow u = \ln \frac{1}{4}, w = 0 \Rightarrow u = 0$$

$$= \frac{1}{2}[e^u]_{\ln(1/4)}^0 = \frac{1}{2}[e^0 - e^{\ln(1/4)}] = \frac{1}{2}\left(1 - \frac{1}{4}\right) = \frac{3}{8}$$

$$25. \int_1^3 \frac{[\ln(v+1)]^2}{v+1} dv = \int_1^3 [\ln(v+1)]^2 \frac{1}{v+1} dv = \int_{\ln 2}^{\ln 4} u^2 du,$$

where $u = \ln(v+1)$, $du = \frac{1}{v+1} dv$; $v = 1 \Rightarrow u = \ln 2$, $v = 3 \Rightarrow u = \ln 4$;

$$= \frac{1}{3}[u^3]_{\ln 2}^{\ln 4} = \frac{1}{3}[(\ln 4)^3 - (\ln 2)^3] = \frac{1}{3}[(2 \ln 2)^3 - (\ln 2)^3] = \frac{(\ln 2)^3}{3}(8 - 1) = \frac{7}{3}(\ln 2)^3$$

$$26. \int_2^4 (1 + \ln t)(t \ln t) dt = \int_2^4 (t \ln t)(1 + \ln t) dt = \int_{2 \ln 2}^{4 \ln 4} u du,$$

where $u = t \ln t$, $du = \left(t \left(\frac{1}{t}\right) + (\ln t)(1)\right) dt = (1 + \ln t) dt$; $t = 2 \Rightarrow u = 2 \ln 2$, $t = 4 \Rightarrow u = 4 \ln 4$,

$$= \frac{1}{2}[u^2]_{2 \ln 2}^{4 \ln 4} = \frac{1}{2}[(4 \ln 4)^2 - (2 \ln 2)^2] = \frac{1}{2}[(8 \ln 2)^2 - (2 \ln 2)^2] = \frac{(2 \ln 2)^2}{2}(16 - 1) = 30(\ln 2)^2$$

$$27. \int_1^8 \frac{\log_4 \theta}{\theta} d\theta = \frac{1}{\ln 4} \int_1^8 (\ln \theta) \left(\frac{1}{\theta}\right) d\theta = \frac{1}{\ln 4} \int_0^{\ln 8} u du, \text{ where } u = \ln \theta, du = \frac{1}{\theta} d\theta, \theta = 1 \Rightarrow u = 0, \theta = 8 \Rightarrow u = \ln 8$$

$$= \frac{1}{2 \ln 4}[u^2]_0^{\ln 8} = \frac{1}{\ln 16}[(\ln 8)^2 - 0^2] = \frac{(3 \ln 2)^2}{4 \ln 2} = \frac{9 \ln 2}{4}$$

$$28. \int_1^e \frac{8(\ln 3)(\log_3 \theta)}{\theta} d\theta = \int_1^e \frac{8(\ln 3)(\ln \theta)}{\theta(\ln 3)} d\theta = 8 \int_1^e (\ln \theta) \left(\frac{1}{\theta}\right) d\theta = 8 \int_0^1 u du,$$

where $u = \ln \theta$, $du = \frac{1}{\theta} d\theta$; $\theta = 1 \Rightarrow u = 0$, $\theta = e \Rightarrow u = 1$,

$$= 4[u^2]_0^1 = 4(1^2 - 0^2) = 4$$

$$29. \int \frac{2 dy}{\sqrt{1+25y^2}} = \frac{2}{5} \int \frac{du}{\sqrt{1+u^2}}, \text{ where } u = 5y \text{ and } du = 5 dy,$$

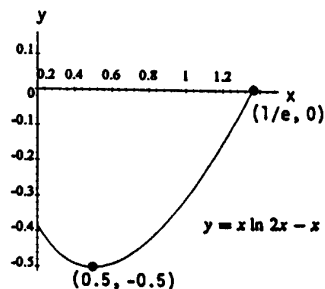
$$= \frac{2}{5} \sin^{-1} u + C = \frac{2}{5} \sin^{-1}(5y) + C$$

$$30. \int \frac{3 \, dy}{4 - 49y^2} = \frac{3}{4} \int \frac{dy}{1 - \frac{49y^2}{4}} = \frac{3}{4} \int \frac{\frac{2}{7} \left(\frac{du}{1-u^2} \right)}{1-u^2} = \frac{3}{14} \int \frac{du}{1-u^2}, \text{ where } u = \frac{7}{2}y \text{ and } du = \frac{7}{2} \, dy,$$

$$= \frac{3}{14} \tanh^{-1} u + C = \frac{3}{14} \tanh^{-1} \left(\frac{7}{2}y \right) + C \text{ for } \left| \frac{7}{2}y \right| < 1 \Rightarrow -\frac{2}{7} < y < \frac{2}{7}$$

$$31. \quad y = x \ln 2x - x \Rightarrow y' = x \left(\frac{2}{2x} \right) + \ln(2x) - 1 = \ln 2x;$$

solving $y' = 0 \Rightarrow x = \frac{1}{2}$; $y' > 0$ for $x > \frac{1}{2}$ and $y' < 0$ for $x < \frac{1}{2} \Rightarrow$ relative minimum of $-\frac{1}{2}$ at $x = \frac{1}{2}$; $f\left(\frac{1}{2e}\right) = -\frac{1}{e}$ and $f\left(\frac{e}{2}\right) = 0 \Rightarrow$ absolute minimum is $-\frac{1}{2}$ at $x = \frac{1}{2}$ and the absolute maximum is 0 at $x = \frac{e}{2}$

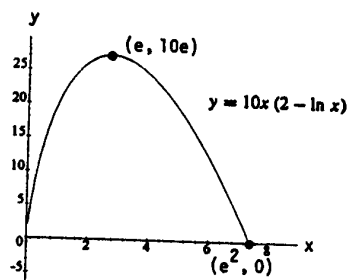


$$32. \quad y = 10x(2 - \ln x) \Rightarrow y' = 10(2 - \ln x) - 10x\left(\frac{1}{x}\right) = 20 - 10 \ln x - 10 = 10(1 - \ln x); \text{ solving } y' = 0$$

$\Rightarrow x = e$; $y' < 0$ for $x > e$ and $y' > 0$ for $x < e$

$\Rightarrow$ relative maximum at $x = e$ of $10e$;

$y(e^2) = 10e^2(2 - 2 \ln e) = 0 \Rightarrow$ absolute minimum is 0 at $x = e^2$ and the absolute maximum is $10e$ at $x = e$



$$33. \quad A = \int_1^e \frac{2 \ln x}{x} \, dx = \int_0^1 2u \, du = [u^2]_0^1 = 1, \text{ where}$$

$u = \ln x$ and $du = \frac{1}{x} \, dx$; $x = 1 \Rightarrow u = 0$, $x = e \Rightarrow u = 1$

$$34. \quad (a) \quad A_1 = \int_{10}^{20} \frac{1}{x} \, dx = [\ln |x|]_{10}^{20} = \ln 20 - \ln 10 = \ln \frac{20}{10} = \ln 2, \text{ and } A_2 = \int_1^2 \frac{1}{x} \, dx = [\ln |x|]_1^2 = \ln 2 - \ln 1 = \ln 2$$

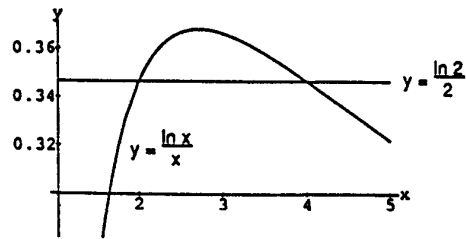
$$(b) \quad A_1 = \int_{ka}^{kb} \frac{1}{x} \, dx = [\ln |x|]_{ka}^{kb} = \ln kb - \ln ka = \ln \frac{kb}{ka} = \ln \frac{b}{a} = \ln b - \ln a, \text{ and } A_2 = \int_a^b \frac{1}{x} \, dx = [\ln |x|]_a^b = \ln b - \ln a$$

$$35. \quad y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}; \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \left(\frac{1}{x} \right) \sqrt{x} = \frac{1}{\sqrt{x}} \Rightarrow \frac{dy}{dt} \Big|_{e^2} = \frac{1}{e} \text{ m/sec}$$

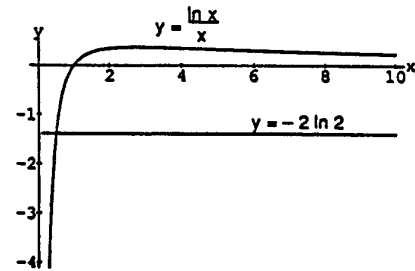
$$36. \quad V = \pi \int_{1/4}^4 \left(\frac{1}{2\sqrt{x}} \right)^2 \, dx = \frac{\pi}{4} \int_{1/4}^4 \frac{1}{x} \, dx = \frac{\pi}{4} [\ln |x|]_{1/4}^4 = \frac{\pi}{4} \left(\ln 4 - \ln \frac{1}{4} \right) = \frac{\pi}{4} \ln 16 = \frac{\pi}{4} \ln(2^4) = \pi \ln 2$$

$$37. \quad \text{The two functions differ by } \ln \frac{5}{3} \text{ because } K = \ln(5x) - \ln(3x) = \ln 5 + \ln x - \ln 3 - \ln x = \ln 5 - \ln 3 = \ln \frac{5}{3}.$$

38. (a) No, there are two intersections: one at $x = 2$
and the other at $x = 4$



- (b) Yes, because there is only one intersection point.



39. Force = Mass times Acceleration (Newton's Second Law) or $F = ma$. Let $a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v \frac{dv}{ds}$. Then

$$\begin{aligned} ma &= -mgR^2s^{-2} \Rightarrow a = -gR^2s^{-2} \Rightarrow v \frac{dv}{ds} = -gR^2s^{-2} \Rightarrow v \, dv = -gR^2s^{-2} \, ds \Rightarrow \int v \, dv = \int -gR^2s^{-2} \, ds \\ \Rightarrow \frac{v^2}{2} &= \frac{gR^2}{s} + C \Rightarrow v^2 = \frac{2gR^2}{s} + 2C_1 = \frac{2gR^2}{s} + C. \text{ When } t = 0, v = v_0 \text{ and } s = R \Rightarrow v_0^2 = \frac{2gR^2}{R} + C \\ \Rightarrow C &= v_0^2 - 2gR \Rightarrow v^2 = \frac{2gR^2}{s} + v_0^2 - 2gR \end{aligned}$$

40. If $v_0 = \sqrt{2gR}$, then $v^2 = \frac{2gR^2}{s} \Rightarrow v = \sqrt{\frac{2gR^2}{s}}$, since $v \geq 0$ if $v_0 \geq \sqrt{2gR}$. Then $\frac{ds}{dt} = \frac{\sqrt{2gR^2}}{\sqrt{s}}$

$$\Rightarrow \sqrt{s} \, ds = \sqrt{2gR^2} \, dt \Rightarrow \int s^{1/2} \, ds = \int \sqrt{2gR^2} \, dt \Rightarrow \frac{2}{3}s^{3/2} = (\sqrt{2gR^2})t + C_1 \Rightarrow s^{3/2} = \left(\frac{3}{2}\sqrt{2gR^2}\right)t + C;$$

$$t = 0 \text{ and } s = R \Rightarrow R^{3/2} = \left(\frac{3}{2}\sqrt{2gR^2}\right)(0) + C \Rightarrow C = R^{3/2} \Rightarrow s^{3/2} = \left(\frac{3}{2}\sqrt{2gR^2}\right)t + R^{3/2}$$

$$= \left(\frac{3}{2}R\sqrt{2g}\right)t + R^{3/2} = R^{3/2} \left[\left(\frac{3}{2}R^{-1/2}\sqrt{2g}\right)t + 1 \right] = R^{3/2} \left[\left(\frac{3\sqrt{2gR}}{2R}\right)t + 1 \right]$$

$$= R^{3/2} \left[\left(\frac{3v_0}{2R}\right)t + 1 \right] \Rightarrow s = R \left[1 + \left(\frac{3v_0}{2R}\right)t \right]^{2/3}$$

41. $y' \cos x - y \sin x = \sin 2x$; Noting that $\frac{d}{dx}(y \cos x) = y' \cos x - y \sin x$, we can rewrite the differential equation

$$\text{as } \frac{d}{dx}(y \cos x) = \sin 2x \Rightarrow y \cos x = -\frac{1}{2} \cos 2x + C \Rightarrow y = \frac{-\cos 2x}{2 \cos x} + \frac{C}{\cos x}; y(0) = 1 \Rightarrow 1 = -\frac{1}{2} + C$$

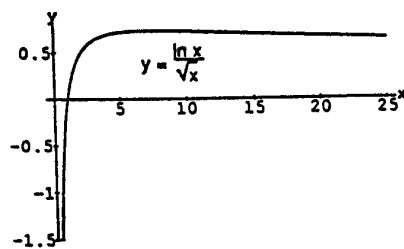
$$\Rightarrow C = \frac{3}{2} \Rightarrow y = \frac{3 - \cos 2x}{2 \cos x}$$

42. $\frac{dy}{dx} = -\frac{y \ln y}{1+x^2} \Rightarrow \frac{dy}{y \ln y} = -\frac{dx}{1+x^2} \Rightarrow \int \frac{\left(\frac{1}{y}\right)}{\ln y} dy = -\int \frac{1}{1+x^2} dx \Rightarrow \ln(\ln y) = -\tan^{-1} x + C$; $x = 0$ and $y = e^2 \Rightarrow \ln(\ln e^2) = -\tan^{-1} 0 + C \Rightarrow \ln 2 = C \Rightarrow \ln(\ln y) = -\tan^{-1} x + \ln 2$
 $\Rightarrow e^{\ln(\ln y)} = e^{(-\tan^{-1} x + \ln 2)} \Rightarrow \ln y = e^{(-\tan^{-1} x + \ln 2)} \Rightarrow e^{\ln y} = e^{e^{(-\tan^{-1} x + \ln 2)}} \Rightarrow y = e^{e^{(-\tan^{-1} x + \ln 2)}}$ or
 $y = \exp(\exp(-\tan^{-1} x + \ln 2))$

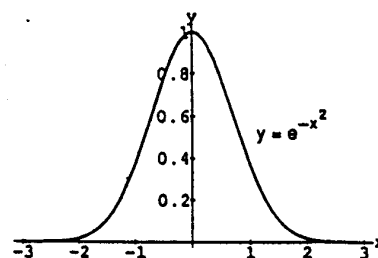
43. $\frac{dy}{dx} + \left(\frac{2}{x+1}\right)y = \frac{x}{x+1} \Rightarrow P(x) = \frac{2}{x+1}, Q(x) = \frac{x}{x+1} \Rightarrow \int P(x) dx = \int \left(\frac{2}{x+1}\right) dx = 2 \ln|x+1| = \ln(x+1)^2$
 $\Rightarrow v(x) = e^{\ln(x+1)^2} = (x+1)^2 \Rightarrow y = \frac{1}{(x+1)^2} \int (x+1)^2 \left(\frac{x}{x+1}\right) dx = \frac{1}{(x+1)^2} \int (x^2 + x) dx$
 $= \left(\frac{1}{x+1}\right)^2 \left(\frac{x^3}{3} + \frac{x^2}{2} + C\right)$; $x = 0$ and $y = 1 \Rightarrow 1 = 0 + 0 + C \Rightarrow C = 1 \Rightarrow y = \frac{1}{(x+1)^2} \cdot \left(\frac{x^3}{3} + \frac{x^2}{2} + 1\right)$

44. $\frac{dy}{dx} + \left(\frac{2}{x}\right)y = \frac{x^2+1}{x} \Rightarrow P(x) = \frac{2}{x}, Q(x) = \frac{x^2+1}{x} \Rightarrow \int P(x) dx = \int \frac{2}{x} dx = 2 \ln|x| = \ln x^2 \Rightarrow v(x) = e^{\ln x^2}$
 $= x^2 \Rightarrow y = \frac{1}{x^2} \int x^2 \left(\frac{x^2+1}{x}\right) dx = \frac{1}{x^2} \int (x^3 + x) dx = \frac{1}{x^2} \left(\frac{x^4}{4} + \frac{x^2}{2} + C\right) = \frac{x^2}{4} + \frac{1}{2} + \frac{C}{x^2}$; $x = 1$ and $y = 1$
 $\Rightarrow 1 = \frac{1}{4} + \frac{1}{2} + C \Rightarrow C = \frac{1}{4} \Rightarrow y = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}$

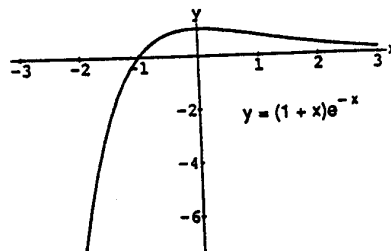
45. (a) $y = \frac{\ln x}{\sqrt{x}} \Rightarrow y' = \frac{1}{x\sqrt{x}} - \frac{\ln x}{2x^{3/2}} = \frac{2 - \ln x}{2x\sqrt{x}}$
 $\Rightarrow y'' = -\frac{3}{4}x^{-5/2}(2 - \ln x) - \frac{1}{2}x^{-5/2} = x^{-5/2}\left(\frac{3}{4}\ln x - 2\right)$;
solving $y' = 0 \Rightarrow \ln x = 2 \Rightarrow x = e^2$; $y' < 0$ for $x > e^2$ and
and $y' > 0$ for $x < e^2 \Rightarrow$ a maximum of $\frac{2}{e}$; $y'' = 0$
 $\Rightarrow \ln x = \frac{8}{3} \Rightarrow x = e^{8/3}$; the curve is concave down on
 $(0, e^{8/3})$ and concave up on $(e^{8/3}, \infty)$



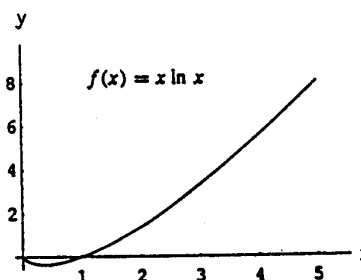
(b) $y = e^{-x^2} \Rightarrow y' = -2xe^{-x^2} \Rightarrow y'' = -2e^{-x^2} + 4x^2e^{-x^2}$;
solving $y' = 0 \Rightarrow x = 0$; $y' < 0$ for $x > 0$ and $y' > 0$ for
 $x < 0 \Rightarrow$ a maximum at $x = 0$ of $e^0 = 1$; there are points
of inflection at $x = \pm \frac{1}{\sqrt{2}}$; the curve is concave down for
 $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and concave up otherwise



- (c) $y = (1+x)e^{-x} \Rightarrow y' = e^{-x} - (1+x)e^{-x} = -xe^{-x}$
 $\Rightarrow y'' = -e^{-x} + xe^{-x} = (x-1)e^{-x}$; solving $y' = 0$
 $\Rightarrow -xe^{-x} = 0 \Rightarrow x = 0$; $y' < 0$ for $x > 0$ and $y' > 0$
for $x < 0 \Rightarrow$ a maximum at $x = 0$ of $(1+0)e^0 = 1$;
there is a point of inflection at $x = 1$ and the curve is
concave up for $x > 1$ and concave down for $x < 1$



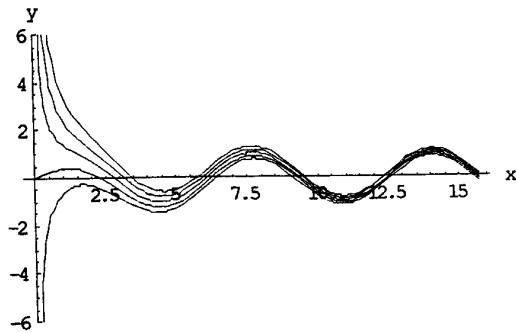
46. $y = x \ln x \Rightarrow y' = \ln x + x \left(\frac{1}{x}\right) = \ln x + 1$; solving $y' = 0$
 $\Rightarrow \ln x + 1 = 0 \Rightarrow \ln x = -1 \Rightarrow x = e^{-1}$; $y' > 0$ for
 $x > e^{-1}$ and $y' < 0$ for $x < e^{-1} \Rightarrow$ a minimum of $e^{-1} \ln e^{-1}$
 $= -\frac{1}{e}$ at $x = e^{-1}$



47. Since the half life is 5700 years and $A(t) = A_0 e^{kt}$ we have $\frac{A_0}{2} = A_0 e^{5700k} \Rightarrow \frac{1}{2} = e^{5700k} \Rightarrow \ln(0.5) = 5700k$
 $\Rightarrow k = \frac{\ln(0.5)}{5700}$. With 10% of the original carbon-14 remaining we have $0.1A_0 = A_0 e^{\frac{\ln(0.5)}{5700}t} \Rightarrow 0.1 = e^{\frac{\ln(0.5)}{5700}t}$
 $\Rightarrow \ln(0.1) = \frac{\ln(0.5)}{5700}t \Rightarrow t = \frac{(5700) \ln(0.1)}{\ln(0.5)} \approx 18,935$ years (rounded to the nearest year).
48. $T - T_s = (T_o - T_s)e^{-kt} \Rightarrow 180 - 40 = (220 - 40)e^{-k/4}$, time in hours, $\Rightarrow k = -4 \ln\left(\frac{7}{9}\right) = 4 \ln\left(\frac{9}{7}\right) \Rightarrow 70 - 40$
 $= (220 - 40)e^{-4 \ln(9/7)t} \Rightarrow t = \frac{\ln 6}{4 \ln(9/7)} \approx 1.78$ hr ≈ 107 min, the total time $\Rightarrow$ the time it took to cool from
 180° F to 70° F was $107 - 15 = 92$ min

49. (a) $xy' + y = x \cos x \Rightarrow y' + \frac{1}{x}y = \cos x \Rightarrow P(x) = \frac{1}{x} \Rightarrow \int P(x) dx = \ln x$ since $x > 0$
 $v(x) = e^{\int P(x) dx} = x \Rightarrow y = \frac{1}{v(x)} \int P(x) \cos x dx = \frac{1}{x} \int x \cos x dx$. Integration by parts: let $u = x$ and
 $dv = \cos x dx$ and $v = \sin x \Rightarrow \int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + C$;
Therefore, $y = \frac{x \sin x + \cos x + C}{x} = \sin x + \frac{C + \cos x}{x}$.

- (b) The graphs, from the bottom curve to the top, are for $C = -2, -1, 0, 1, 2$, respectively.



As $x \rightarrow \infty$, all of the solution curves get closer to $\sin x$ regardless of the value for C . As $x \rightarrow 0^+$, the solution curves get close to $\frac{C+1}{x}$ if $C \neq -1$, and if $C = -1$, then $\lim_{x \rightarrow 0^+} y(x) = 0$.

50. Use the Fundamental Theorem of Calculus.

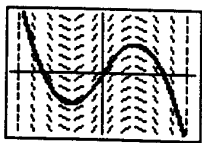
$$y' = \frac{d}{dx} \left(\int_0^x \sin t^2 \, dt \right) + \frac{d}{dx} (x^3 + x + 2) = (\sin x^2) = (3x^2 + 1)$$

$$y'' = \frac{d}{dx} (\sin x^2 + 3x^2 + 1) = (\cos x^2)(2x) + 6x = 2x \cos(x^2) + 6x$$

Thus, the differential equation is satisfied. Verify the initial conditions: $y'(0) = (\sin 0^2) + 3(0)^2 + 1 = 1$,

$$y(0) = \int_0^0 \sin(t^2) \, dt + 0^3 + 0 + 2 = 2$$

- 51.



$[-10, 10]$ by $[-10, 10]$

52. To find the approximate values let $y_n = y_{n-1} + (y_{n-1} + \cos x_{n-1})(0.1)$ with $x_0 = 0$, $y_0 = 0$, and 20 steps. Use a spreadsheet, graphing calculator, or CAS to obtain the values in the following table.

x	y	x	y
0	0	1.1	1.6241
0.1	0.1000	1.2	1.8319
0.2	0.2095	1.3	2.0513
0.3	0.3285	1.4	2.2832
0.4	0.4568	1.5	2.5285
0.5	0.5946	1.6	2.7884
0.6	0.7418	1.7	3.0643
0.7	0.8986	1.8	3.3579
0.8	1.0649	1.9	3.6709
0.9	1.2411	2.0	4.0057
1.0	1.4273		

53. To find the approximate solution let $z_n = y_{n-1} + ((2 - y_{n-1})(2x_{n-1} + 3))(0.1)$ and

$$y_n = y_{n-1} + \left(\frac{(2 - y_{n-1})(2x_{n-1} + 3) + (2 - z_n)(2x_n + 3)}{2} \right)(0.1) \text{ with initial values } x_0 = -3, y_0 = 1,$$

and 20 steps. Use a spreadsheet, graphing calculator, or CAS to obtain the values in the following table.

x	y	x	y
-3	1	-1.9	-5.9686
-2.9	0.6680	-1.8	-6.5456
-2.8	0.2599	-1.7	-6.9831
-2.7	-0.2294	-1.6	-7.2562
-2.6	-0.8011	-1.5	-7.3488
-2.5	-1.4509	-1.4	-7.2553
-2.4	-2.1687	-1.3	-6.9813
-2.3	-2.9374	-1.2	-6.5430
-2.2	-3.7333	-1.1	-5.9655
-2.1	-4.5268	-1.0	-5.2805
-2.0	-5.2840		

54. To estimate $y(3)$, let $z_n = y_{n-1} + \left(\frac{x_{n-1} - 2y_{n-1}}{x_{n-1} + 1} \right)(0.05)$ and $y_n = y_{n-1} + \frac{1}{2} \left(\frac{x_{n-1} - 2y_{n-1}}{x_{n-1} + 1} + \frac{x_n - 2z_n}{x_n + 1} \right)(0.05)$ with initial values $x_0 = 0, y_0 = 1$, and 60 steps. Use a spreadsheet, programmable calculator, or CAS to obtain $y(3) \approx 0.9063$.

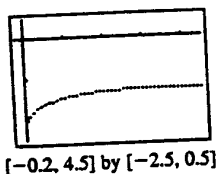
55. To estimate $y(4)$, let $y_n = y_{n-1} + \left(\frac{x_{n-1}^2 - 2y_{n-1} + 1}{x_{n-1}} \right)(0.05)$ with initial values $x_0 = 1, y_0 = 1$, and 60 steps.

Use a spreadsheet, programmable calculator, or CAS to obtain $y(4) \approx 4.4974$.

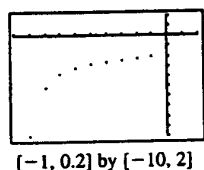
56. Let $y_n = y_{n-1} + \left(\frac{1}{e^{x_{n-1} + y_{n-1} + 2}} \right)(dx)$ with starting values $x_0 = 0$ and $y_0 = -2$, and steps of 0.1 and -0.1 .

Use a spreadsheet, programmable calculator, or CAS to generate the following graphs.

(a)

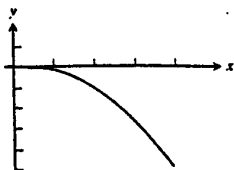


(b) Note that we choose a small interval of x -values because the y -values decrease very rapidly and our calculator cannot handle the calculations for $x \leq -1$. (This occurs because the analytic solution is $y = -2 + \ln(2 - e^{-x})$, which has an asymptote at $x = -\ln 2 \approx -0.69$. Obviously, the Euler approximations are misleading for $x \leq -0.7$.)

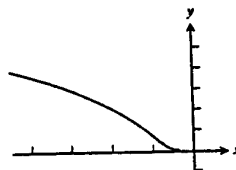


57. Let $z_n = y_{n-1} - \left(\frac{x_{n-1}^2 + y_{n-1}}{e^{y_{n-1} + x_{n-1}}} \right) (dx)$ and $y_n = y_{n-1} + \frac{1}{2} \left(\frac{x_{n-1}^2 + y_{n-1}}{e^{y_{n-1} + x_{n-1}}} + \frac{x_n^2 + z_n}{e^{z_n + x_n}} \right) (dx)$ with starting values $x_0 = 0, y_0 = 0$, and steps of 0.1 and -0.1. Use a spreadsheet, programmable calculator, or CAS to generate the following graphs.

(a)



(b)



$$\begin{aligned}
 58. \quad (a) \quad \frac{dP}{dt} &= 0.002P \left(1 - \frac{P}{800} \right) \Rightarrow \frac{dP}{dt} = 0.002P \left(\frac{800-P}{800} \right) \Rightarrow \frac{800}{P(800-P)} dP = 0.002 dt \Rightarrow \frac{(800-P)+P}{P(800-P)} = 0.002 dt \\
 &\Rightarrow \int \left(\frac{1}{P} + \frac{1}{800-P} \right) dP = 0.002 dt \Rightarrow \ln|P| - \ln|800-P| = 0.002t + C \Rightarrow \ln \left| \frac{P}{800-P} \right| = 0.002t + C \\
 &\Rightarrow \ln \left| \frac{800-P}{P} \right| = -0.002t - C \Rightarrow \left| \frac{800-P}{P} \right| = e^{-0.002t-C} \Rightarrow \frac{800-P}{P} = \pm e^{-C} e^{-0.002t} \\
 &\Rightarrow \frac{800}{P} - 1 = Ae^{-0.002t} \Rightarrow P = \frac{800}{1 + Ae^{-0.002t}}
 \end{aligned}$$

$$\text{Initial condition: } P(0) = 50 \Rightarrow 50 = \frac{800}{1 + Ae^0} \Rightarrow 1 + A = 16 \Rightarrow A = 15$$

$$\text{Solution: } P = \frac{800}{1 + 15e^{-0.002t}}$$

$$(b) \quad \frac{dP}{dt} = 0.002P \left(1 - \frac{P}{800} \right), P(0) = 50 \Rightarrow P_{n+1} = P_n + 0.002P_n \left(1 - \frac{P_n}{800} \right) dt = P_n + 0.001P_n \left(1 - \frac{P_n}{800} \right)$$

On a TI-02 Plus calculator home screen, type the following commands:

50 STO> p:0 STO> t:p (enter)

p+0.001*p*(1-p/800) STO> p:t+0.5 STO> t:p (enter, 40 times)

The last value displayed gives $P_{\text{Euler}}(20) \approx 51.9073$

$$\text{From part (a), } P_{\text{exact}}(20) = \frac{800}{1 + 15e^{-0.002(20)}} \approx 51.9081 \Rightarrow \left| \frac{P_{\text{Euler}}(20) - P_{\text{exact}}(20)}{P_{\text{exact}}(20)} \right| \times 100\%$$

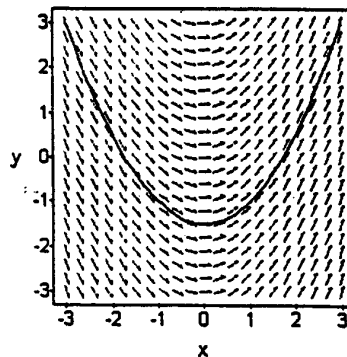
$$= \left| \frac{51.9073 - 51.9081}{51.9081} \right| \times 100\% \approx 0.154\%$$

$$59. \quad \begin{array}{c|cccccc} x & 1 & 1.2 & 1.4 & 1.6 & 1.8 & 2.0 \\ \hline y & -1 & -0.8 & -0.56 & -0.28 & 0.04 & 0.4 \end{array}$$

$$\frac{dy}{dx} = x \Rightarrow dy = x dx \Rightarrow y = \frac{x^2}{2} + C; x = 1 \text{ and } y = -1$$

$$\Rightarrow -1 = \frac{1}{2} + C \Rightarrow C = -\frac{3}{2} \Rightarrow y \text{ (exact)} = \frac{x^2}{2} - \frac{3}{2}$$

$$\Rightarrow y(2) = \frac{2^2}{2} - \frac{3}{2} = \frac{1}{2} \text{ is the exact value}$$

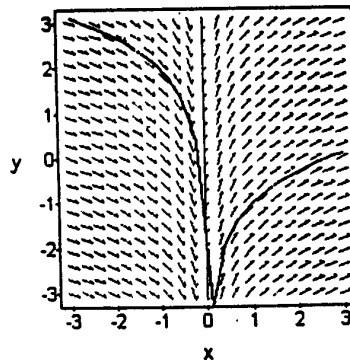


60.	x	1	1.2	1.4	1.6	1.8	2.0
	y	-1	-0.8	-0.6333	-0.4904	-0.3654	-0.2544

$$\frac{dy}{dx} = \frac{1}{x} \Rightarrow dy = \frac{1}{x} dx \Rightarrow y = \ln|x| + C; x = 1 \text{ and } y = -1$$

$$\Rightarrow -1 = \ln 1 + C \Rightarrow C = -1 \Rightarrow y \text{ (exact)} = \ln|x| - 1$$

$$\Rightarrow y(2) = \ln 2 - 1 \approx -0.3069 \text{ is the exact value}$$



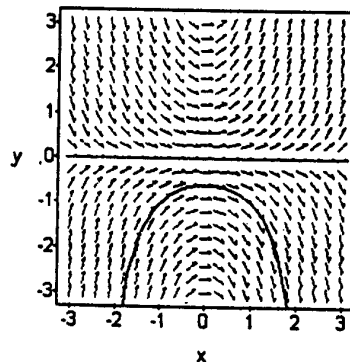
61.	x	1	1.2	1.4	1.6	1.8	2.0
	y	-1	-1.2	-1.488	-1.9046	-2.5141	-3.4192

$$\frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = x dx \Rightarrow \ln|y| = \frac{x^2}{2} + C \Rightarrow y = e^{\frac{x^2}{2} + C} = e^{x^2/2} \cdot e^C$$

$$= C_1 e^{x^2/2}; x = 1 \text{ and } y = -1 \Rightarrow -1 = C_1 e^{1/2} \Rightarrow C_1 = -e^{-1/2}$$

$$\Rightarrow y \text{ (exact)} = -e^{-1/2} \cdot e^{x^2/2} = -e^{(x^2-1)/2} \Rightarrow y(2) = -e^{3/2}$$

$$\approx -4.4817 \text{ is the exact value}$$

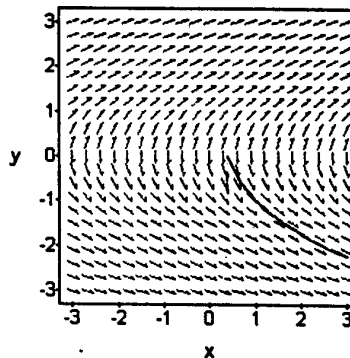


62.	x	1	1.2	1.4	1.6	1.8	2.0
	y	-1	-1.2	-1.3667	-1.5130	-1.6452	-1.7668

$$\frac{dy}{dx} = \frac{1}{y} \Rightarrow y dy = dx \Rightarrow \frac{y^2}{2} = x + C; x = 1 \text{ and } y = -1$$

$$\Rightarrow \frac{1}{2} = 1 + C \Rightarrow C = -\frac{1}{2} \Rightarrow y^2 = 2x - 1 \Rightarrow y \text{ (exact)} = -\sqrt{2x-1}$$

$$\Rightarrow y(2) = -\sqrt{3} \approx -1.7321 \text{ is the exact value}$$



CHAPTER 6 ADDITIONAL EXERCISES—THEORY, EXAMPLES, APPLICATIONS

$$1. A_1 = \int_1^e \frac{2 \log_2 x}{x} dx = \frac{2}{\ln 2} \int_1^e \frac{\ln x}{x} dx = \left[\frac{(\ln x)^2}{\ln 2} \right]_1^e = \frac{1}{\ln 2}; A_2 = \int_1^e \frac{2 \log_4 x}{4} dx = \frac{2}{\ln 4} \int_1^e \frac{\ln x}{x} dx$$

$$= \left[\frac{(\ln x)^2}{2 \ln 2} \right]_1^e = \frac{1}{2 \ln 2} \Rightarrow A_1 : A_2 = 2 : 1$$

$$2. f(x) = e^{g(x)} \Rightarrow f'(x) = e^{g(x)} g'(x), \text{ where } g'(x) = \frac{x}{1+x^4} \Rightarrow f'(2) = e^0 \left(\frac{2}{1+16} \right) = \frac{2}{17}$$

3. (a) $\frac{df}{dx} = \frac{2 \ln e^x}{e^x} \cdot e^x = 2x$

(b) $f(0) = \int_1^1 \frac{2 \ln t}{t} dt = 0$

(c) $\frac{df}{dx} = 2x \Rightarrow f(x) = x^2 + C$; $f(0) = 0 \Rightarrow C = 0 \Rightarrow f(x) = x^2 \Rightarrow$ the graph of $f(x)$ is a parabola

4. (a) The figure shows that $\frac{\ln e}{e} > \frac{\ln \pi}{\pi} \Rightarrow \pi \ln e > e \ln \pi \Rightarrow \ln e^\pi > \ln \pi^e \Rightarrow e^\pi > \pi^e$

(b) $y = \frac{\ln x}{x} \Rightarrow y' = \left(\frac{1}{x}\right)\left(\frac{1}{x}\right) - \frac{\ln x}{x^2} \Rightarrow \frac{1 - \ln x}{x^2}$; solving $y' = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$; $y' < 0$ for $x > e$ and $y' > 0$ for $0 < x < e \Rightarrow$ an absolute maximum occurs at $x = e$

5. The area of the shaded region is $\int_0^1 \sin^{-1} x \, dx = \int_0^1 \sin^{-1} y \, dy$, which is the same as the area of the region to the left of the curve $y = \sin x$ (and part of the rectangle formed by the coordinate axes and dashed lines $y = 1$,

$x = \frac{\pi}{2}$). The area of the rectangle is $\frac{\pi}{2} = \int_0^1 \sin^{-1} y \, dy + \int_0^{\pi/2} \sin x \, dx$, so we have

$$\frac{\pi}{2} = \int_0^1 \sin^{-1} x \, dx + \int_0^{\pi/2} \sin x \, dx \Rightarrow \int_0^{\pi/2} \sin x \, dx = \frac{\pi}{2} - \int_0^1 \sin^{-1} x \, dx.$$

6. (a) slope of $L_3 < \text{slope of } L_2 < \text{slope of } L_1 \Rightarrow \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$

(b) area of small (shaded) rectangle $<$ area under curve $<$ area of large rectangle

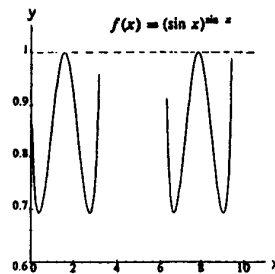
$$\Rightarrow \frac{1}{b}(b - a) < \int_a^b \frac{1}{x} \, dx < \frac{1}{a}(b - a) \Rightarrow \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$$

7. Method 1: Use a CAS or a numerical integral function on a calculator or spreadsheet to define $y_1 = x^2 \ln x$ and $y_2 = \frac{d}{dx} \left(\frac{x^3 \ln x}{3} - \frac{x^3}{9} + C \right)$, then compare the graph of y_1 with that of y_2 . The graphs should be the same.

Method 2: Use a CAS or a numerical integration function on a calculator or spreadsheet to define

$y_1 = \int_a^x t^2 \ln t \, dt$ and $y_2 = \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C$ (you pick an $a > 0$ and any value for C), then compare the graph of y_1 with that of y_2 . The graphs should be the same except for a vertical translation.

8. In the interval $\pi < x < 2\pi$ the function $\sin x < 0$
 $\Rightarrow (\sin x)^{\sin x}$ is not defined for all values in that interval or its translation by 2π .



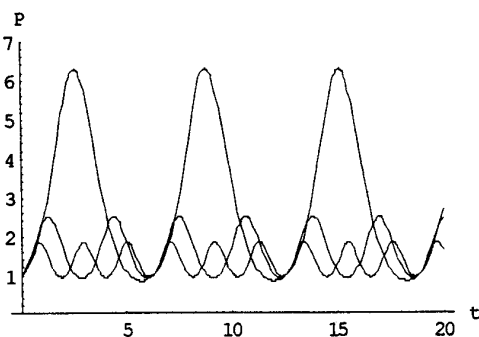
9. (a) $\frac{dP}{dt} = kP \cos(at - b) \Rightarrow \frac{dP}{P} = k \cos(at - b) dt \Rightarrow \ln|P| = \frac{k}{a} \sin(at - b) + C$

$$\Rightarrow |P| = e^{\frac{k}{a} \sin(at - b) + C_1} = e^{C_1} e^{\frac{k}{a} \sin(at - b)} = C_2 e^{\frac{k}{a} \sin(at - b)} \Rightarrow P = \pm C_2 e^{\frac{k}{a} \sin(at - b)} = C e^{\frac{k}{a} \sin(at - b)} \text{ where}$$

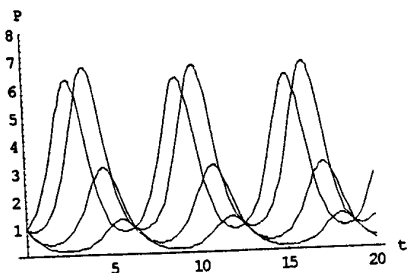
$$C_2 = e^{C_1} \text{ and } C = \pm C_2; P(0) = P_0 \Rightarrow C e^{-\frac{k}{a} \sin b} \Rightarrow C = P_0 e^{\frac{k}{a} \sin b} \Rightarrow P = P_0 e^{\frac{k}{a} \sin b} e^{\frac{k}{a} \sin(at - b)}$$

$$= P_0 e^{\frac{k}{a} (\sin b + \sin(at - b))}$$

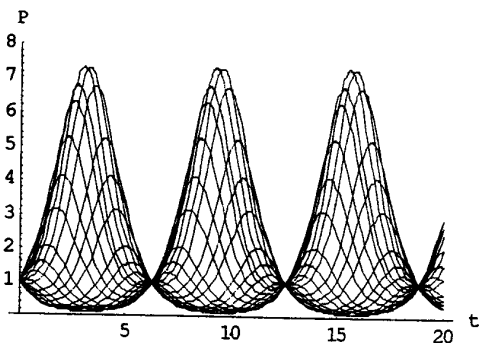
(b) The following graphs are for $a = 1, 2, 3, b = 1, k = 1$



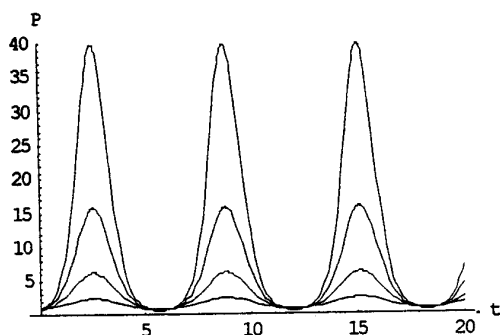
The graphs show that as a increases, the frequency of the oscillations increases and the amplitude decreases. The next graphs are for $a = 1, b = 1, 2, 3, 4, k = 1$



The graphs show that as b increases the solution curve shifts to the right and the amplitude oscillates. An interesting picture is obtained by graphing the solution curves for $a = 1, b = 1, 2, \dots, 20, k = 1$ and is shown in the next graph.



The next graphs are for $a = 1, b = 1, k = 0.5, 1.0, 1.5, 2.0$



The graphs show that as k increases, the maximum population increases and the minimum population decreases, getting closer to zero. In the long term, all the population appears to oscillate for all values of the parameters.

$$\begin{aligned}
 10. \quad (a) \quad \frac{dy}{dt} &= k \frac{A}{V}(c-y) \Rightarrow dy = -k \frac{A}{V}(y-c) dt \Rightarrow \frac{dy}{y-c} = -k \frac{A}{V} dt \Rightarrow \int \frac{dy}{y-c} = - \int k \frac{A}{V} dt \Rightarrow \ln|y-c| \\
 &= -k \frac{A}{V} t + C_1 \Rightarrow y-c = \pm e^{C_1} e^{-k \frac{A}{V} t}. \text{ Apply the initial condition, } y(0) = y_0 \Rightarrow y_0 = c + C \Rightarrow C = y_0 - c \\
 &\Rightarrow y = c + (y_0 - c) e^{-k \frac{A}{V} t}
 \end{aligned}$$

$$(b) \text{ Steady state solution: } y_\infty = \lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} \left[c + (y_0 - c) e^{-k \frac{A}{V} t} \right] = c + (y_0 - c)(0) = c$$

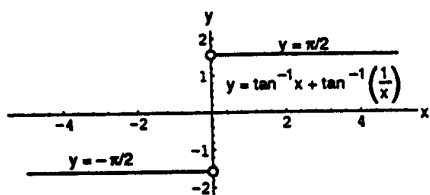
$$11. \quad y = \tan^{-1} x + \tan^{-1} \left(\frac{1}{x} \right) \Rightarrow y' = \frac{1}{1+x^2} + \frac{\left(-\frac{1}{x^2} \right)}{\left(1 + \frac{1}{x^2} \right)}$$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0 \Rightarrow \tan^{-1} x + \tan^{-1} \left(\frac{1}{x} \right) \text{ is a constant}$$

and the constant is $\frac{\pi}{2}$ for $x > 0$; it is $-\frac{\pi}{2}$ for $x < 0$ since

$$\tan^{-1} x + \tan^{-1} \left(\frac{1}{x} \right) \text{ is odd. Next the } \lim_{x \rightarrow 0^+} \left[\tan^{-1} x + \tan^{-1} \left(\frac{1}{x} \right) \right]$$

$$= 0 + \frac{\pi}{2} = \frac{\pi}{2} \text{ and } \lim_{x \rightarrow 0^-} \left(\tan^{-1} x + \tan^{-1} \left(\frac{1}{x} \right) \right) = 0 + \left(-\frac{\pi}{2} \right) = -\frac{\pi}{2}$$



NOTES: